

GEOMETRIC ANALYSIS OF REDUCTIONS FROM SCHLESINGER TRANSFORMATIONS TO DIFFERENCE PAINLEVÉ EQUATIONS

ANTON DZHAMAY AND TOMOYUKI TAKENAWA

ABSTRACT. We present two examples of reductions from the evolution equations describing discrete Schlesinger transformations of Fuchsian systems to difference Painlevé equations: difference Painlevé equation $d\text{-}P\left(A_2^{(1)*}\right)$ with the symmetry group $E_6^{(1)}$ and difference Painlevé equation $d\text{-}P\left(A_1^{(1)*}\right)$ with the symmetry group $E_7^{(1)}$. In both cases we describe in detail how to compute their Okamoto space of the initial conditions and emphasize the role played by geometry in helping us to understand the structure of the reduction, a choice of a good coordinate system describing the equation, and how to compare it with other instances of equations of the same type.

1. INTRODUCTION

It is well-known that many special functions, e.g. the Airy, Bessel, or Legendre functions, originate as series solutions of *linear* ordinary differential equations (ODEs). Such functions play a crucial role in describing a wide range of important physical and mathematical phenomena. An essential point here is that for linear ordinary differential equations singularities of solutions can only occur at the points where the coefficients of the equation itself become singular. This makes it possible to talk about global properties of solutions (and hence, of the corresponding special functions), and the asymptotic behavior of solutions near those fixed singular points.

For nonlinear equations the situation is very different. Although the Cauchy existence theorem guarantees *local* existence of the solution to a given initial value problem at an ordinary (regular) point, in general the domain in which this solution is defined depends not just on the equation itself, but on the initial conditions as well — solutions acquire *movable* (i.e., dependent on the initial values or, equivalently, on the constants of integration) singularities. Such singularities are called *critical* if a solution loses its single-valued character in a neighborhood of the singularity (e.g., when a singular point is a *branch point*). An ODE is said to satisfy the *Painlevé property* if its general solution is free of *movable* critical singular points. Otherwise, the Riemann surface uniformizing such solution becomes dependent on the constants of integration, which prevents global analysis. Thus, equivalently, the Painlevé property of an ODE is the uniformizability of its general solution, see [Con99a] (as well as the other excellent articles in the volume [Con99b]) for a careful overview of these ideas.

It is clear that linear equations satisfy Painlevé property. The importance of the Painlevé property for nonlinear equations is that, similar to the linear case, solutions of these equations give rise to new transcendental functions. In that sense, according to M.Kruskal as quoted in [GR14], nonlinear equations satisfying this property are on the border between trivially integrable linear equations and nonlinear equations that are not integrable, and so the Painlevé property is essentially equivalent to (and is a criterion of) integrability.

The search for new transcendental functions was the original motivation in the work of P.Painlevé who, together with his student B.Gambier, had classified all of the rational second-order differential equations that have the Painlevé property, [Pai02], [Pai73], [Gam10]. Among 50 classes of equations that they found, only *six* can not be reduced to linear equations or integrated by quadratures. These equations are now known as $P_I - P_{VI}$. Solutions to these equations, the so-called *Painlevé transcendents*, are playing an increasingly important role in describing a wide range of nonlinear phenomena in mathematics and physics [IKSY91].

2010 *Mathematics Subject Classification.* 34M55, 34M56, 14E07.

Key words and phrases. Integrable systems, Painlevé equations, difference equations, isomonodromic transformations, birational transformations.

Almost simultaneously with the work of P.Painlevé and B.Gambier, the most general Painlevé VI equation was obtained by R.Fuchs [Fuc05] in the theory of *isomonodromic deformations* of Fuchsian systems. This theory, developed in the works of R.Fuchs [Fuc07], L.Schlesinger [Sch12], R.Garnier [Gar26], and then extended to the non-Fuchsian case by M.Jimbo, T.Miwa, and K.Ueno [JMU81, JM82], and also by H.Flaschka and A.Newell [FN80], as well as the related Riemann-Hilbert approach [IN86], [FIKN06], are now among the most powerful methods for studying the structure of the Painlevé transcendents.

Over the last thirty years a significant effort has been put towards understanding and generalizing results and methods of the classical theory of completely integrable systems to the *discrete* case. This is true for the theory of Painlevé equations as well. Discrete Painlevé equations were originally defined as second order nonlinear difference equations that have usual Painlevé equations as continuous limits [BK90], [GM90]. A systematic study of discrete Painlevé equations was started by B.Grammaticos, J.Hietarinta, F.Nijhoff, V.Papageorgiou and A.Ramani, [NP91], [RGH91], [GRP91], and many different examples of such equations were obtained in a series of papers by Grammaticos, Ramani, and their collaborators by a systematic application of the singularity confinement criterion, see reviews [GR04], [GR14], and many references therein. Discrete Painlevé equations also appear in a broad spectrum of important nonlinear problems in mathematics and physics, among which are the theory of orthogonal polynomials, quantum gravity, determinantal random point processes, reductions of integrable lattice equations, and, notably, as Bäcklund transformations of differential Painlevé equations. Some of these problems are discrete analogues or direct discretizations of the corresponding nonlinear problems, and some describe purely discrete phenomena.

It turned out that classifying discrete Painlevé equations by their continuous limits, as well as the singularity confinement criterion, is not a very good approach, since such correspondence is far from being bijective. It is both possible for the same discrete equation to have different continuous Painlevé equations as continuous limits under different limiting procedures, and for different discrete equations to have the same continuous limit. In the seminal paper [Sak01] H.Sakai showed that an effective way to understand and classify discrete Painlevé equations is through algebraic geometry. In this approach, to each equation, if we consider it as a two-dimensional first-order nonlinear system, we put in correspondence a family of algebraic surfaces $\mathcal{X}_{\mathbf{b}}$, where $\mathbf{b} = \{b_i\}$ is some collection of parameters that change, depending on the type of the equation, in an additive, multiplicative, or elliptic fashion as functions of a discrete “time” parameter. This family $\mathcal{X}_{\mathbf{b}}$, by a slight abuse of terminology, is called the *Okamoto space of initial conditions* (that we often denote simply by \mathcal{X} omitting explicit dependence on parameters b_i) and it is obtained by resolving the indeterminacy points of the corresponding map $\varphi : \mathbb{P}^2 \rightarrow \mathbb{P}^2$ via the blowing-up procedure. By Sakai’s theory, the complete resolution of indeterminacies is obtained by blowing up 9 (possibly infinitely close) points on \mathbb{P}^2 (or eight points on a birationally equivalent $\mathbb{P}^1 \times \mathbb{P}^1$ compactification of \mathbb{C}^2). The resulting surface \mathcal{X} has a special property that it admits a unique anti-canonical divisor $D \in |-K_{\mathcal{X}}|$ of *canonical type*. The orthogonal complement of $-K_{\mathcal{X}}$ in the Picard lattice $\text{Pic}(\mathcal{X}) \simeq H^2(\mathcal{X}; \mathbb{Z})$ is described by the affine Dynkin diagram $E_8^{(1)}$ that has two intersecting root subsystems of affine type: R , that is generated by classes D_i of irreducible components of the anticanonical divisor D , and its orthogonal complement R^{\perp} whose corresponding root lattice $Q(R^{\perp})$ is called the *symmetry sub-lattice*. Then the *type* of the discrete Painlevé equation is the same as the *type of its surface* $\mathcal{X}_{\mathbf{b}}$, which is just the type of an affine Dynkin diagram describing the root system R of irreducible components D_i of D (essentially, the degeneration structure of the positions of the blowup points). Moreover, nonlinear Painlevé dynamic now becomes a translation in the symmetry sub-lattice $Q(R^{\perp})$, and so it becomes “linearized” there, see [Sak01] for details. This is somewhat similar to the algebro-geometric integrability of classical integrable systems and soliton equations, where nonlinear dynamic is mapped to commuting linear flows on the Jacobian of the spectral curve of the associated linear problem via the Abel-Jacobi map.

One important observation from Sakai’s geometric approach is that in addition to *additive* (difference) and *multiplicative* (q -difference) *discretizations* of continuous Painlevé equations, there are some purely discrete Painlevé equations. It also led to the discovery of the master *elliptic* discrete Painlevé equation such that all of the other Painlevé equations can be obtained from it through degenerations (which corresponds to more and more special configurations of the blow-up points). This degenerations can be described by the following scheme, where letters stand for the symmetry type of the equation (which is the type of the affine Dynkin diagram of the root subsystem R^{\perp}), and the subscripts e , q , δ , and c stand for elliptic, multiplicative, additive and differential Painlevé equations respectively, see Figure 1.

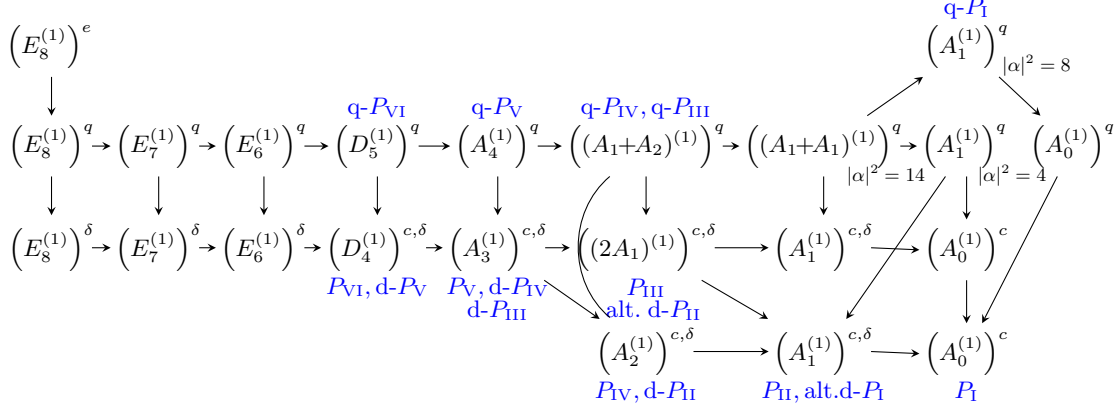


FIGURE 1. Inclusion scheme for the symmetry sub-lattices and corresponding Painlevé equations

The following question then becomes natural and important: *How to represent these new purely discrete equations in the isomonodromic framework?* This question was posed by Sakai in [Sak07] (**Problem A** for the difference case and **Problems B,C** for the q -difference case).

More precisely, in both continuous and discrete difference case we start with some Fuchsian system and consider its isomonodromic deformations. In the continuous case, deformation parameters are locations of singular points of the system. The resulting isomonodromic flows on the space of coefficients of the Fuchsian system are given by Schlesinger *equations*. In particular, for a 2×2 Fuchsian system with four poles, Schlesinger equations reduce to the most general P_{VI} equation. In the discrete difference case, deformation parameters are the characteristic indices of the system and since the isomonodromy condition requires that the indices change by integral shifts, the resulting dynamic is indeed *discrete*. It is expressed in the form of *difference* equations called Schlesinger *transformations*. It is also possible to get the isomonodromic description of difference and q -difference Painlevé equations by studying deformations of linear difference and q -difference analogues of Fuchsian systems, see [JS96] and [Bor04] for details.

In [DST13] we studied in detail a particular class of Schlesinger transformations that are called *elementary*. These transformations change only two of the characteristic indices of the underlying Fuchsian system (and any other Schlesinger transformation not involving characteristic indices with multiplicity can be represented as a composition of the elementary ones). In particular, we obtained explicit evolution equations governing the resulting discrete dynamic. Our objective for the present paper is to carefully and very explicitly describe reductions of these *discrete Schlesinger evolution equations* to the *difference* Painlevé equations.

Since the Painlevé equations are of second order, we focus on Fuchsian systems that have two-dimensional moduli spaces (coordinates on such moduli space are known as accessory parameters). It turns out that, modulo two natural transformations called *Katz's addition* and *middle convolution* [Kat96], that preserve the deformation equations [HF07], there are only four such systems, [Kos01], that have the *spectral type* $(11, 11, 11, 11)$, $(111, 111, 111)$, $(22, 1111, 1111)$, and $(33, 222, 111111)$ (spectral type of a Fuchsian system encodes the degeneracies of the characteristic indices, or eigenvalues of residue matrices at singular points, and it is carefully defined in the next section).

Isomonodromic deformations of a $(11, 11, 11, 11)$ spectral type Fuchsian system are well known — continuous deformations reduce to Painlevé VI equation and Schlesinger transformations reduce to the difference Painlevé $d-P(D_4^{(1)})$ equation, also known as $d-P_V$, and in [DST13] we showed that in this case our *discrete Schlesinger evolution equations* indeed can be reduced to the standard form of $d-P(D_4^{(1)})$.

In [Boa09] P.Boalch showed that for Fuchsian systems with the spectral types $(111, 111, 111)$, $(22, 1111, 1111)$, and $(33, 222, 111111)$ their Schlesinger transformations reduce to difference Painlevé equations with the required symmetry groups $E_6^{(1)}$, $E_7^{(1)}$, and $E_8^{(1)}$ respectively, thus providing a theoretical answer to Sakai's **Problem A**.

Our goal for the present paper is to make this statement very concrete via explicit direct computation of reductions of discrete Schlesinger evolution to difference Painlevé equations with symmetry groups $E_6^{(1)}$ and $E_7^{(1)}$ (we plan to consider deformations of a Fuchsian system of the spectral type (33, 222, 111111) with the symmetry group $E_8^{(1)}$ elsewhere). In addition to establishing that the resulting difference Painlevé equations have the required types $d-P(A_2^{(1)*})$ and $d-P(A_1^{(1)*})$, we explicitly compare the resulting equations with the previously known instances of equations of the same type. We do so by finding an explicit identification between their Okamoto spaces of initial conditions, which allows us to compute and compare the translation directions for different equations w.r.t. the same root basis, and also to match generic parameters to the characteristic indices of the Fuchsian system, which in turn allows us to see and compare these different equations via their actions on the Riemann scheme of our Fuchsian system. We show that in both examples elementary Schlesinger dynamic is indeed more elementary in the sense that standard examples of difference Painlevé equations can be realized as compositions of elementary Schlesinger transformations. Of particular interest here is the $E_7^{(1)}$ case which has two characteristic indices of multiplicity 2. We show that in that case the standard form of the equation can not be represented as a composition of elementary Schlesinger transformations of rank one. Thus we first generalize our discrete Schlesinger evolution equations from [DST13] to elementary Schlesinger transformations of rank two, and then show how to represent the standard dynamic as a composition of two such rank two transformations. We also provide a lot of details on how to compute the Okamoto space of initial conditions for our equations and how to identify two different instances of such spaces, hoping that this will be helpful for other researchers who are interested in the geometric approach to discrete Painlevé equations.

The paper is organized as follows. In Section 2 we briefly describe our parameterization of a Fuchsian system by its spectral and eigensystem data, define elementary Schlesinger transformations, present evolution equations for elementary Schlesinger transformations as a dynamic on the space of coefficient of our Fuchsian system, and then show how to split them to get the dynamic on the space of eigenvectors of the coefficient matrices (this is a brief overview of our paper [DST13]). Next we show how to generalize this to elementary Schlesinger transformations of rank two, which is a new result. In Section 3 we consider two examples of reductions of the elementary Schlesinger transformation dynamic. The first example of a difference Painlevé equation of type $d-P(\tilde{A}_2^*)$ with the symmetry group \tilde{E}_6 was also briefly presented in [DST13], here we go into a lot more detail and show how the choice of good coordinates, which was essentially guessed in [DST13], is really forced on us by the geometric considerations. The next example of a 4×4 Fuchsian system of the *spectral type* 22, 1111, 1111 (i.e., with three poles and two double eigenvalues at one pole) is a first example that we have which has degenerate eigenvalues, and this is a completely new result. Finally, we give a brief summary in Section 4.

Acknowledgements. Part of this work was done when A.D. was visiting T.T. at the Tokyo University of Marine Science and Technology and Nalini Joshi at the University of Sydney, and A.D. would like to thank both Universities for the stimulating working environment and, together with the University of Northern Colorado, for the generous travel support.

2. PRELIMINARIES

The goal of this section is to write down evolution equations for elementary Schlesinger transformations, as well as to introduce the necessary notation. Our presentation here is very brief and we refer the interested reader to [DST13] for details. The main new and important result of this section is the generalization of equations governing elementary Schlesinger transformation dynamic on the decomposition space from rank-one to rank-two Schlesinger transformations.

2.1. Fuchsian Systems. Consider a generic *Fuchsian system* (or a *Fuchsian equation*) written in the *Schlesinger normal form*:

$$(2.1) \quad \frac{d\mathbf{Y}}{dz} = \mathbf{A}(z)\mathbf{Y} = \left(\sum_{i=1}^n \frac{\mathbf{A}_i}{z - z_i} \right) \mathbf{Y}, \quad z_i \neq z_j \text{ for } i \neq j,$$

where $\mathbf{A}_i = \text{res}_{z_i} \mathbf{A}(z) dz$ are constant $m \times m$ matrices. In addition to simple poles at finite distinct points z_1, \dots, z_n , this system also has a simple pole at $z_0 = \infty \in \mathbb{P}^1$ if $\mathbf{A}_\infty = \text{res}_\infty \mathbf{A}(z) dz = -\sum_{i=1}^n \mathbf{A}_i \neq \mathbf{0}$. The *spectral data* of system (2.1) consists of locations of the simple poles z_1, \dots, z_n and the eigenvalues (also called the *characteristic indices*) θ_i^j of \mathbf{A}_i and their multiplicities. These multiplicities are encoded by the *spectral type* of the system,

$$\mathbf{m} = m_1^1 \cdots m_1^{l_1}, m_2^1 \cdots m_2^{l_2}, \dots, m_n^1 \cdots m_n^{l_n}, m_\infty^1 \cdots m_\infty^{l_\infty},$$

where partitions $m = m_i^1 + \cdots + m_i^{l_i}$, $m_i^1 \geq \cdots \geq m_i^{l_i} \geq 1$ describe the multiplicities of the eigenvalues of \mathbf{A}_i . Spectral type classifies Fuchsian systems up to isomorphisms and the operations of addition and middle convolution.

2.2. Schlesinger Transformations. Schlesinger transformations are discrete analogues of the usual Schlesinger differential equations describing isomonodromic deformations of our Fuchsian system. They are rational transformations preserving the singularity structure and the monodromy data of the system (2.1), except for the integral shifts in the characteristic indices θ_i^j , and so the coefficient matrix now depends on θ_i^j , $\mathbf{A} = \mathbf{A}(z; \Theta)$. Schlesinger transformations are given by the following *differential-difference Lax Pair*:

$$\begin{cases} \frac{d\mathbf{Y}}{dz} = \mathbf{A}(z; \Theta)\mathbf{Y} = \left(\sum_{i=1}^n \frac{\mathbf{A}_i(\Theta)}{z - z_i} \right) \mathbf{Y}, \\ \bar{\mathbf{Y}}(z) = \mathbf{R}(z)\mathbf{Y}(z). \end{cases},$$

where $\mathbf{R}(z)$ is a specially chosen rational matrix function called the *multiplier* of the transformation. The compatibility condition of this Lax Pair is

$$(2.2) \quad \bar{\mathbf{A}}(z; \Theta)\mathbf{R}(z) = \mathbf{R}(z)\mathbf{A}(z; \Theta) + \frac{d\mathbf{R}(z)}{dz}.$$

In [DST13] we considered a special class of Schlesinger transformations for which the multiplier matrix has the form

$$(2.3) \quad \mathbf{R}(z) = \mathbf{I} + \frac{z_0 - \zeta_0}{z - z_0} \mathbf{P}, \quad \text{where } \mathbf{P} = \mathbf{P}^2 \text{ is a projector.}$$

It turns out that in this case it is possible to solve equation (2.2) explicitly to obtain a discrete dynamic on the space of coefficient matrices. Namely, after substituting $\mathbf{R}(z)$ of the form (2.3) in (2.2) (and its inverse) we immediately see that $z_0, \zeta_0 \in \{z_i\}_{i=1}^n$, and if we put $z_0 = z_\alpha$ and $\zeta_0 = z_\beta$, we get the following equations on the coefficient matrices:

$$(2.4) \quad \begin{aligned} \bar{\mathbf{A}}_i &= \mathbf{R}(z_i)\mathbf{A}_i\mathbf{R}^{-1}(z_i) \quad \text{for } i \neq \alpha, \beta \quad (\text{and therefore } \bar{\Theta}_i = \Theta_i), \\ \mathbf{Q}\bar{\mathbf{A}}_\alpha &= \mathbf{A}_\alpha\mathbf{Q}, \quad \bar{\mathbf{A}}_\beta\mathbf{Q} = \mathbf{Q}\mathbf{A}_\beta, \end{aligned}$$

$$(2.5) \quad \begin{aligned} \bar{\mathbf{A}}_\alpha\mathbf{P} &= \mathbf{P}\mathbf{A}_\alpha - \mathbf{P}, \quad \mathbf{P}\bar{\mathbf{A}}_\beta = \mathbf{A}_\beta\mathbf{P} + \mathbf{P}, \\ \bar{\mathbf{A}}_\alpha &= \mathbf{A}_\alpha + \sum_{i \neq \alpha} \left(\frac{z_\alpha - z_\beta}{z - z_\alpha} \right) (\mathbf{P}\mathbf{A}_i - \bar{\mathbf{A}}_i\mathbf{P}), \\ \bar{\mathbf{A}}_\beta &= \mathbf{A}_\beta + \sum_{i \neq \beta} \left(\frac{z_\beta - z_\alpha}{z - z_\beta} \right) (\mathbf{A}_i\mathbf{P} - \mathbf{P}\bar{\mathbf{A}}_i). \end{aligned}$$

Then either (2.4) or (2.5) imposes important constraints on the projector \mathbf{P} :

$$(2.6) \quad \mathbf{P}\mathbf{A}_\alpha\mathbf{Q} = \mathbf{0} \quad (\text{or } \mathbf{P}\mathbf{A}_\alpha\mathbf{P} = \mathbf{P}\mathbf{A}_\alpha), \quad \mathbf{Q}\mathbf{A}_\beta\mathbf{P} = \mathbf{0} \quad (\text{or } \mathbf{P}\mathbf{A}_\beta\mathbf{P} = \mathbf{A}_\beta\mathbf{P}),$$

and if this condition is satisfied, we get the following dynamic on the space of coefficient matrices:

$$(2.7) \quad \bar{\mathbf{A}}_i = \mathbf{R}(z_i) \mathbf{A}_i \mathbf{R}^{-1}(z_i), \quad i \neq \alpha, \beta,$$

$$(2.8) \quad \bar{\mathbf{A}}_\alpha = \mathbf{A}_\alpha - \mathbf{Q} \mathbf{A}_\alpha \mathbf{P} - \mathbf{P} + \sum_{i \neq \alpha} \left(\frac{z_\beta - z_\alpha}{z_i - z_\alpha} \right) \mathbf{P} \mathbf{A}_i \mathbf{Q},$$

$$(2.9) \quad \bar{\mathbf{A}}_\beta = \mathbf{A}_\beta - \mathbf{P} \mathbf{A}_\beta \mathbf{Q} + \mathbf{P} + \sum_{i \neq \beta} \left(\frac{z_\alpha - z_\beta}{z_i - z_\beta} \right) \mathbf{Q} \mathbf{A}_i \mathbf{P}.$$

Indeed,

$$\begin{aligned} \bar{\mathbf{A}}_\alpha &= \bar{\mathbf{A}}_\alpha \mathbf{P} + \bar{\mathbf{A}}_\alpha \mathbf{Q} = \mathbf{P} \mathbf{A}_\alpha - \mathbf{P} + \mathbf{A}_\alpha (\mathbf{I} - \mathbf{P}) + \sum_{i \neq \alpha} \left(\frac{z_\alpha - z_\beta}{z_\alpha - z_i} \right) \mathbf{P} \mathbf{A}_i \mathbf{Q} \\ &= \mathbf{A}_\alpha - \mathbf{Q} \mathbf{A}_\alpha \mathbf{P} - \mathbf{P} + \sum_{i \neq \alpha} \left(\frac{z_\beta - z_\alpha}{z_i - z_\alpha} \right) \mathbf{P} \mathbf{A}_i \mathbf{Q}, \end{aligned}$$

and the equation for $\bar{\mathbf{A}}_\beta$ is obtained in a similar fashion. We call these equations *Discrete Schlesinger Evolution Equations*.

2.3. The Decomposition Space. It is sometimes more convenient to extend the dynamic from the space of coefficients of the Fuchsian system to the space of eigenvectors of the coefficient matrices, we call this space the *decomposition space*. In particular, this is the space on which both the continuous ([JMMS80]) and the discrete ([DST13]) Hamiltonian equations for Schlesinger deformations can be written. Before defining this space it is convenient to reduce the number of parameters in our system by using scalar gauge transformations of the form $\tilde{\mathbf{Y}}(z) = w(z)^{-1} \mathbf{Y}(z)$, where $w(z)$ is a solution of the *scalar equation*

$$\frac{dw}{dz} = \sum_{i=1}^n \frac{\theta_i^j}{z - z_i} w.$$

Such transformations change the residue matrices by $\tilde{\mathbf{A}}_i = \mathbf{A}_i - \theta_i^j \mathbf{I}$ (and consequently change the residue matrix at infinity by $\tilde{\mathbf{A}}_\infty = \mathbf{A}_\infty + \theta_i^j \mathbf{I}$). Hence we can always make one of the eigenvalues $\theta_i^j = 0$ by choosing a good representative w.r.t. the action by the group of local *scalar gauge transformations*. So we make the following assumption.

Assumption 2.1. We always assume that at the finite point z_i the eigenvalue θ_i^1 of the highest multiplicity m_i^1 is zero.

We also need the following important *semi-simplicity* assumption.

Assumption 2.2. We assume that the coefficient matrices \mathbf{A}_i are *diagonalizable* (even when we have multiple eigenvalues).

In view of these assumptions, coefficient matrices \mathbf{A}_i are similar to diagonal matrices $\text{diag}\{\theta_i^1, \dots, \theta_i^{r_i}, 0, \dots, 0\}$, where $r_i = \text{rank}(\mathbf{A}_i)$. Omitting the zero eigenvalues, we put

$$(2.10) \quad \Theta_i = \text{diag}\{\theta_i^1, \dots, \theta_i^{r_i}\}.$$

Further, in view of Assumption (2.2) there exist full sets of *right* eigenvectors $\mathbf{b}_{i,j}$, $\mathbf{A}_i \mathbf{b}_{i,j} = \theta_i^j \mathbf{b}_{i,j}$, and *left* eigenvectors $\mathbf{c}_i^{j\dagger}$, $\mathbf{c}_i^{j\dagger} \mathbf{A}_i = \theta_i^j \mathbf{c}_i^{j\dagger}$ (we use the \dagger symbol to indicate a *row*-vector or a matrix of row vectors). In the matrix form, omitting vectors with indices $j > r_i$ that are in the kernel of \mathbf{A}_i , we can write

$$\mathbf{B}_i = [\mathbf{b}_{i,1} \cdots \mathbf{b}_{i,r_i}], \quad \mathbf{A}_i \mathbf{B}_i = \mathbf{B}_i \Theta_i, \quad \mathbf{C}_i^\dagger = \begin{bmatrix} \mathbf{c}_i^{1\dagger} \\ \vdots \\ \mathbf{c}_i^{r_i\dagger} \end{bmatrix}, \quad \mathbf{C}_i^\dagger \mathbf{A}_i = \Theta_i \mathbf{C}_i^\dagger,$$

with Θ_i defined by (2.10). Then we have a decomposition $\mathbf{A}_i = \mathbf{B}_i \mathbf{C}_i^\dagger$, provided that $\mathbf{C}_i^\dagger \mathbf{B}_i = \Theta_i$. We call this last condition the *orthogonality condition* (since Θ_i is diagonal) and we assume that it holds even when we have repeating eigenvalues. This condition is related to the normalization ambiguity of the eigenvectors. Thus, given \mathbf{A}_i , we can construct (in a non-unique way) a corresponding decomposition pair $(\mathbf{B}_i, \mathbf{C}_i^\dagger)$. The

space of all such pairs for all finite indices $1 \leq i \leq n$, without any additional conditions, is our *decomposition space*. We denote it as

$$\begin{aligned}\mathcal{B} \times \mathcal{C} &= (\mathbb{C}^{r_1} \times \cdots \times \mathbb{C}^{r_n}) \times ((\mathbb{C}^{r_1})^\dagger \times \cdots \times (\mathbb{C}^{r_n})^\dagger) \\ &\simeq (\mathbb{C}^{r_1} \times (\mathbb{C}^{r_1})^\dagger) \times \cdots \times (\mathbb{C}^{r_n} \times (\mathbb{C}^{r_n})^\dagger)\end{aligned}$$

and write an element $(\mathbf{B}, \mathbf{C}^\dagger)$ of this space as a list of n pairs $(\mathbf{B}_1, \mathbf{C}_1^\dagger; \cdots; \mathbf{B}_n, \mathbf{C}_n^\dagger)$. Then, given a Riemann Scheme of a Fuchsian system (equivalently, a collection $\Theta = \{\theta_i^j\}$ of the characteristic indices having the correct multiplicities and satisfying the Fuchs relation), we denote by

$$(2.11) \quad (\mathcal{B} \times \mathcal{C})_\Theta = \{(\mathbf{B}_1, \mathbf{C}_1^\dagger; \cdots; \mathbf{B}_n, \mathbf{C}_n^\dagger) \in \mathcal{B} \times \mathcal{C} \mid \mathbf{C}_i^\dagger \mathbf{B}_i = \Theta_i, \sum_{i=1}^n \mathbf{B}_i \mathbf{C}_i^\dagger = \mathbf{A}_\infty \sim \Theta_\infty\}$$

the corresponding fiber in the decomposition space (since for Schlesinger transformations locations of the poles are just fixed parameters of the dynamic, we occasionally omit them, as in the above notation).

Remark 2.3. There are two natural actions on the decomposition space $\mathcal{B} \times \mathcal{C}$. First, the group \mathbb{GL}_m of global gauge transformations of the Fuchsian system induces the following action. Given $\mathbf{P} \in \mathbb{GL}_m$, we have the action $\mathbf{A}_i \mapsto \mathbf{P} \mathbf{A}_i \mathbf{P}^{-1}$ which translates into the action $(\mathbf{B}_i, \mathbf{C}_i^\dagger) \mapsto (\mathbf{P} \mathbf{B}_i, \mathbf{C}_i^\dagger \mathbf{P}^{-1})$. We refer to such transformations as *similarity transformations*. It is often necessary to restrict this action to the subgroup $G_{\mathbf{A}_\infty}$ preserving the form of \mathbf{A}_∞ . Second, for any pair $(\mathbf{B}_i, \mathbf{C}_i^\dagger)$ the pair $(\mathbf{B}_i \mathbf{Q}_i, \mathbf{Q}_i^{-1} \mathbf{C}_i^\dagger)$ determines the same matrix \mathbf{A}_i for $\mathbf{Q}_i \in \mathbb{GL}_{r_i}$. The condition $\mathbf{Q}_i^{-1} \mathbf{C}_i^\dagger \mathbf{B}_i \mathbf{Q}_i = \mathbf{Q}_i^{-1} \Theta_i \mathbf{Q}_i = \Theta_i$ restricts \mathbf{Q}_i to the stabilizer subgroup G_{Θ_i} of \mathbb{GL}_{r_i} . In particular, when all θ_i^j are distinct, \mathbf{Q}_i has to be a diagonal matrix. We refer to such transformations as *trivial transformations*. These two actions obviously commute with each other. The phase space for the Schlesinger dynamic is the quotient space of $(\mathcal{B} \times \mathcal{C})_\Theta$ by this action.

2.4. Schlesinger Dynamic on the Decomposition Space. In this section we explain how to lift the Schlesinger Evolution equations to the decomposition space.

2.4.1. Rank One. In [DST13] we focused on the *elementary* Schlesinger transformations $\left\{ \begin{smallmatrix} \alpha & \beta \\ \mu & \nu \end{smallmatrix} \right\}$ that only change two of the characteristic indices by unit shifts, i.e., $\bar{\theta}_\alpha^\mu = \theta_\alpha^\mu - 1$ and $\bar{\theta}_\beta^\nu = \theta_\beta^\nu + 1$, $\alpha \neq \beta$. For such transformations the projector matrix \mathbf{P} has rank one and the multiplier matrix has the form (2.3) with

$$(2.12) \quad \mathbf{R}(z) = \mathbf{I} + \frac{z_\alpha - z_\beta}{z - z_\alpha} \mathbf{P}, \quad \text{where } \mathbf{P} = \frac{\mathbf{b}_{\beta, \nu} \mathbf{c}_\alpha^{\mu \dagger}}{\mathbf{c}_\alpha^{\mu \dagger} \mathbf{b}_{\beta, \nu}}, \quad \text{and we put } \mathbf{Q} = \mathbf{I} - \mathbf{P}.$$

In this case, under the semi-simplicity Assumption (2.2) it is possible to decompose equations (2.7–2.9) to get the dynamic on the space $(\mathcal{B} \times \mathcal{C})_\Theta$.

Theorem 2.4 ([DST13]). *An elementary Schlesinger transformation $\left\{ \begin{smallmatrix} \alpha & \beta \\ \mu & \nu \end{smallmatrix} \right\}$ defines the map*

$$(\mathcal{B} \times \mathcal{C})_\Theta \rightarrow (\bar{\mathcal{B}} \times \bar{\mathcal{C}})_{\bar{\Theta}}$$

given by the following evolution equations (grouped for convenience), where c_i^j are arbitrary non-zero constants.

(i) *Transformation vectors:*

$$(2.13) \quad \bar{\mathbf{b}}_{\alpha, \mu} = \frac{1}{c_\alpha^\mu} \mathbf{b}_{\beta, \nu}, \quad \bar{\mathbf{c}}_\beta^{\nu \dagger} = c_\beta^\nu \mathbf{c}_\alpha^{\mu \dagger}.$$

(ii) *Generic indices:*

$$(2.14) \quad \bar{\mathbf{b}}_{i, j} = \frac{1}{c_i^j} \mathbf{R}(z_i) \mathbf{b}_{i, j}, \quad (i \neq \alpha \text{ and if } i = \beta, j \neq \nu);$$

$$(2.15) \quad \bar{\mathbf{c}}_i^{j \dagger} = c_i^j \mathbf{c}_i^{j \dagger} \mathbf{R}^{-1}(z_i), \quad (i \neq \beta \text{ and if } i = \alpha, j \neq \mu).$$

(iii) *Special indices:*

$$(2.16) \quad \bar{\mathbf{b}}_{\alpha,j} = \frac{1}{c_{\alpha}^j} \left(\mathbf{I} - \frac{\mathbf{P}}{\theta_{\alpha}^{\mu} - \theta_{\alpha}^j - 1} \left(\sum_{i \neq \alpha} \frac{z_{\beta} - z_{\alpha}}{z_i - z_{\alpha}} \mathbf{A}_i \right) \right) \mathbf{b}_{\alpha,j}, \quad j \neq \mu;$$

$$(2.17) \quad \bar{\mathbf{c}}_{\beta}^{j\dagger} = c_{\beta}^j \mathbf{c}_{\beta}^{j\dagger} \left(\mathbf{I} - \left(\sum_{i \neq \beta} \frac{z_{\alpha} - z_{\beta}}{z_i - z_{\beta}} \mathbf{A}_i \right) \frac{\mathbf{P}}{\theta_{\beta}^{\nu} - \theta_{\beta}^j + 1} \right), \quad j \neq \nu;$$

$$(2.18) \quad \bar{\mathbf{b}}_{\beta,\nu} = \frac{1}{c_{\beta}^{\nu}} \left((\theta_{\beta}^{\nu} + 1) \mathbf{I} + \mathbf{Q} \left(\mathbf{I} + \sum_{j \neq \nu} \frac{\mathbf{b}_{\beta,j} \mathbf{c}_{\beta}^{j\dagger}}{\theta_{\beta}^{\nu} - \theta_{\beta}^j + 1} \right) \left(\sum_{i \neq \beta} \frac{z_{\alpha} - z_{\beta}}{z_i - z_{\beta}} \mathbf{A}_i \right) \right) \frac{\mathbf{b}_{\beta,\nu}}{\mathbf{c}_{\alpha}^{\mu\dagger} \mathbf{b}_{\beta,\nu}};$$

$$(2.19) \quad \bar{\mathbf{c}}_{\alpha}^{\mu\dagger} = c_{\alpha}^{\mu} \frac{\mathbf{c}_{\alpha}^{\mu\dagger}}{\mathbf{c}_{\alpha}^{\mu\dagger} \mathbf{b}_{\beta,\nu}} \left((\theta_{\alpha}^{\mu} - 1) \mathbf{I} + \left(\sum_{i \neq \alpha} \frac{z_{\beta} - z_{\alpha}}{z_i - z_{\alpha}} \mathbf{A}_i \right) \left(\mathbf{I} + \sum_{j \neq \mu} \frac{\mathbf{b}_{\alpha,j} \mathbf{c}_{\alpha}^{j\dagger}}{\theta_{\alpha}^{\mu} - \theta_{\alpha}^j - 1} \right) \mathbf{Q} \right).$$

2.4.2. *Rank Two.* For the difference Painlevé equation $d-P(A_1^{(1)*})$ we need to study Schlesinger transformations of a Fuchsian system that has the spectral type 22, 1111, 1111, and so we need to consider Schlesinger transformations that change not one but two eigenvalues at each point z_{α} and z_{β} . In this section we show how to obtain the corresponding dynamic on the decomposition space. The resulting equations suggest what happens in the general case of a projector \mathbf{P} of arbitrary rank, but since the focus of the present paper is on examples, we plan to consider the general case elsewhere.

Naively, we want to consider Schlesinger transformations of the form

$$(2.20) \quad \left\{ \begin{smallmatrix} \alpha & \beta \\ \mu_1 & \nu_1 \end{smallmatrix} \right\} = \left\{ \begin{smallmatrix} \alpha & \beta \\ \mu_1 & \nu_1 \end{smallmatrix} \right\} \circ \left\{ \begin{smallmatrix} \alpha & \beta \\ \mu_2 & \nu_2 \end{smallmatrix} \right\} = \left\{ \begin{smallmatrix} \alpha & \beta \\ \mu_2 & \nu_2 \end{smallmatrix} \right\} \circ \left\{ \begin{smallmatrix} \alpha & \beta \\ \mu_1 & \nu_1 \end{smallmatrix} \right\}.$$

However, if one of the characteristic indices (say, α) has multiplicity, applying a rank-one elementary Schlesinger transformation will change the spectral type of the equation (e.g., in our example, a rank-one transformation $\left\{ \begin{smallmatrix} 1 & 2 \\ 1 & 1 \end{smallmatrix} \right\}$ maps the moduli space of Fuchsian equations of spectral type (22, 1111, 1111) to a smaller moduli space (112, 1111, 1111), and in fact our formulas in this case do not work, since some of the expressions become undefined). Thus, we need to develop the rank-two version of the elementary Schlesinger transformation separately. We start with a composition of two rank-one maps to get an insight on the structure of the multiplier matrix in the rank-two case, but then proceed to derive the dynamic equations independently. The resulting equations are then defined on moduli spaces of Fuchsian systems that have multiplicity in the spectral type (e.g., in our example, the map $\left\{ \begin{smallmatrix} 1 & 2 \\ 1 & 1 \end{smallmatrix} \right\}$ is defined on both moduli spaces of Fuchsian systems of spectral type 1111, 1111, 1111 and 22, 1111, 1111). So we start with the multiplier matrix that is a product (and for simplicity we put $\mu_i = i$ and $\nu_j = j$ for this derivation)

$$\mathbf{R}(z) = \bar{\mathbf{R}}_1(z) \mathbf{R}_2(z) = \left(\mathbf{I} + \frac{z_{\alpha} - z_{\beta}}{z - z_{\alpha}} \bar{\mathbf{P}}_1 \right) \left(\mathbf{I} + \frac{z_{\alpha} - z_{\beta}}{z - z_{\alpha}} \mathbf{P}_2 \right),$$

where, in view of (2.12) and (2.14-2.15),

$$\mathbf{P}_i = \frac{\mathbf{b}_{\beta,i} \mathbf{c}_{\alpha}^{i\dagger}}{\mathbf{c}_{\alpha}^{i\dagger} \mathbf{b}_{\beta,i}}, \quad \text{and} \quad \bar{\mathbf{P}}_1 = \frac{\bar{\mathbf{b}}_{\beta,1} \mathbf{c}_{\alpha}^{1\dagger}}{\mathbf{c}_{\alpha}^{1\dagger} \bar{\mathbf{b}}_{\beta,1}} = \frac{\mathbf{Q}_2 \mathbf{b}_{\beta,1} \mathbf{c}_{\alpha}^{1\dagger} \mathbf{Q}_2}{\mathbf{c}_{\alpha}^{1\dagger} \mathbf{Q}_2 \mathbf{b}_{\beta,1}} = \frac{\mathbf{Q}_2 \mathbf{P}_1 \mathbf{Q}_2}{\text{Tr}(\mathbf{Q}_2 \mathbf{P}_1)}.$$

Here $\mathbf{Q}_i = \mathbf{I} - \mathbf{P}_i$ is, as usual, the complementary projector. Then, since clearly $\bar{\mathbf{P}}_1 \mathbf{P}_2 = \mathbf{0}$,

$$\mathbf{R}(z) = \mathbf{I} + \frac{z_{\alpha} - z_{\beta}}{z - z_{\alpha}} \mathcal{P}, \quad \text{where } \mathcal{P} = \bar{\mathbf{P}}_1 + \mathbf{P}_2 \text{ is also a projector.}$$

Let us now rewrite \mathcal{P} in a more symmetric form. First note that, since \mathbf{P}_i are rank-one projectors,

$$\text{Tr}(\mathbf{Q}_2\mathbf{P}_1) = \text{Tr}(\mathbf{P}_1 - \mathbf{P}_2\mathbf{P}_1) = 1 - \text{Tr}(\mathbf{P}_1\mathbf{P}_2) = \text{Tr}(\mathbf{Q}_1\mathbf{P}_2).$$

Also, note that for any rank-one projector \mathbf{S} and for any matrix \mathbf{M} we have an identity $\mathbf{SMS} = \text{Tr}(\mathbf{MS})\mathbf{S}$. Therefore,

$$\mathcal{P} = \frac{\mathbf{Q}_2\mathbf{P}_1\mathbf{Q}_2 + \mathbf{P}_2\mathbf{Q}_1\mathbf{P}_2}{\text{Tr}(\mathbf{Q}_2\mathbf{P}_1)} = \frac{\mathbf{Q}_2\mathbf{P}_1}{\text{Tr}(\mathbf{Q}_2\mathbf{P}_1)} + \frac{\mathbf{Q}_1\mathbf{P}_2}{\text{Tr}(\mathbf{Q}_1\mathbf{P}_2)} = \mathcal{P}_1 + \mathcal{P}_2,$$

where

$$(2.21) \quad \mathcal{P}_1 = \frac{\mathbf{Q}_2\mathbf{P}_1}{\text{Tr}(\mathbf{Q}_2\mathbf{P}_1)} = \frac{\mathbf{Q}_2\mathbf{b}_{\beta,1}\mathbf{c}_\alpha^{1\dagger}}{\mathbf{c}_\alpha^{1\dagger}\mathbf{Q}_2\mathbf{b}_{\beta,1}}, \quad \mathcal{P}_2 = \frac{\mathbf{Q}_1\mathbf{P}_2}{\text{Tr}(\mathbf{Q}_1\mathbf{P}_2)} = \frac{\mathbf{Q}_1\mathbf{b}_{\beta,2}\mathbf{c}_\alpha^{2\dagger}}{\mathbf{c}_\alpha^{2\dagger}\mathbf{Q}_1\mathbf{b}_{\beta,2}}$$

are two mutually orthogonal rank-one projectors, $\mathcal{P}_i^2 = \mathcal{P}_i$, $\mathcal{P}_1\mathcal{P}_2 = \mathcal{P}_2\mathcal{P}_1 = \mathbf{0}$. At the same time, since obviously $\mathbf{Q}_2\mathbf{P}_1 + \mathbf{Q}_1\mathbf{P}_2 = \mathbf{P}_1\mathbf{Q}_2 + \mathbf{P}_2\mathbf{Q}_1$, $\mathcal{P} = \tilde{\mathcal{P}}_1 + \tilde{\mathcal{P}}_2$, where

$$(2.22) \quad \tilde{\mathcal{P}}_1 = \frac{\mathbf{P}_1\mathbf{Q}_2}{\text{Tr}(\mathbf{Q}_2\mathbf{P}_1)} = \frac{\mathbf{b}_{\beta,1}\mathbf{c}_\alpha^{1\dagger}\mathbf{Q}_2}{\mathbf{c}_\alpha^{1\dagger}\mathbf{Q}_2\mathbf{b}_{\beta,1}}, \quad \tilde{\mathcal{P}}_2 = \frac{\mathbf{P}_2\mathbf{Q}_1}{\text{Tr}(\mathbf{Q}_1\mathbf{P}_2)} = \frac{\mathbf{b}_{\beta,2}\mathbf{c}_\alpha^{2\dagger}\mathbf{Q}_1}{\mathbf{c}_\alpha^{2\dagger}\mathbf{Q}_1\mathbf{b}_{\beta,2}}.$$

We also put $\mathcal{Q}_i = \mathbf{I} - \mathcal{P}_i$, $\tilde{\mathcal{Q}}_i = \mathbf{I} - \tilde{\mathcal{P}}_i$, and $\mathcal{Q} = \mathbf{I} - \mathcal{P}$. In view of the orthogonality conditions $\mathbf{C}_i^\dagger\mathbf{B}_i = \mathbf{\Theta}_i$, it is easy to describe the eigenvectors for each of those projectors (we do it just for \mathcal{P} s since for \mathcal{Q} s eigenvectors are the same but eigenvalues swap between 0 and 1, below we use the notation $(\theta, \mathbf{w}^\dagger, \mathbf{v})$, where θ is an eigenvalue (which is either 0 or 1 for projectors), \mathbf{w}^\dagger is a row (or left) eigenvector and \mathbf{v} is a column (or right) eigenvector):

$$(2.23) \quad \text{Eigen}(\mathbf{P}_i) = \{(1; \mathbf{c}_\alpha^{i\dagger}, \mathbf{b}_{\beta,i}), (0; \mathbf{c}_\beta^{j\dagger}, \mathbf{b}_{\alpha,j}) \text{ for } j \neq i\}, \quad i = 1, 2;$$

$$(2.24) \quad \text{Eigen}(\mathcal{P}_1) = \{(1; \mathbf{c}_\alpha^{1\dagger}, \mathbf{Q}_2\mathbf{b}_{\beta,1}), (0; \mathbf{c}_\beta^{2\dagger}, \mathbf{b}_{\alpha,2}), (0; \mathbf{c}_\beta^{j\dagger}, \mathbf{b}_{\alpha,j}) \text{ for } j > 2\};$$

$$(2.25) \quad \text{Eigen}(\mathcal{P}_2) = \{(1; \mathbf{c}_\alpha^{2\dagger}, \mathbf{Q}_1\mathbf{b}_{\beta,2}), (0; \mathbf{c}_\alpha^{1\dagger}, \mathbf{b}_{\alpha,1}), (0; \mathbf{c}_\beta^{j\dagger}, \mathbf{b}_{\alpha,j}) \text{ for } j > 2\};$$

$$(2.26) \quad \text{Eigen}(\tilde{\mathcal{P}}_1) = \{(1; \mathbf{c}_\alpha^{1\dagger}\mathbf{Q}_2, \mathbf{b}_{\beta,1}), (0; \mathbf{c}_\beta^{2\dagger}, \mathbf{b}_{\beta,2}), (0; \mathbf{c}_\beta^{j\dagger}, \mathbf{b}_{\alpha,j}) \text{ for } j > 2\};$$

$$(2.27) \quad \text{Eigen}(\tilde{\mathcal{P}}_2) = \{(1; \mathbf{c}_\alpha^{2\dagger}\mathbf{Q}_1, \mathbf{b}_{\beta,2}), (0; \mathbf{c}_\beta^{1\dagger}, \mathbf{b}_{\beta,1}), (0; \mathbf{c}_\beta^{j\dagger}, \mathbf{b}_{\alpha,j}) \text{ for } j > 2\};$$

$$(2.28) \quad \text{Eigen}(\mathcal{P}) = \{(1; \mathbf{c}_\alpha^{1\dagger}, \mathbf{Q}_2\mathbf{b}_{\beta,1}), (1; \mathbf{c}_\alpha^{2\dagger}, \mathbf{Q}_1\mathbf{b}_{\beta,2}), (0; \mathbf{c}_\beta^{j\dagger}, \mathbf{b}_{\alpha,j}) \text{ for } j > 2\}$$

$$(2.29) \quad = \{(1; \mathbf{c}_\alpha^{1\dagger}\mathbf{Q}_2, \mathbf{b}_{\beta,1}), (1; \mathbf{c}_\alpha^{2\dagger}\mathbf{Q}_1, \mathbf{b}_{\beta,2}), (0; \mathbf{c}_\beta^{j\dagger}, \mathbf{b}_{\alpha,j}) \text{ for } j > 2\}.$$

Remark 2.5. Note that the sum of two rank-one projectors is not a projector. Here \mathcal{P} is the “correct” way to add \mathbf{P}_1 and \mathbf{P}_2 so that the result is a rank-two projector that is a sum of two orthogonal rank-one projectors and that has the same row and column spaces as $\mathbf{P}_1 + \mathbf{P}_2$. Also, note that there are many ways to choose bases in the row and column ranges of \mathcal{P} , the choices above reflect the splittings $\mathcal{P} = \mathcal{P}_1 + \mathcal{P}_2 = \tilde{\mathcal{P}}_1 + \tilde{\mathcal{P}}_2$.

We can now use these projectors to split the discrete Schlesinger evolution equations to define dynamic on eigenvectors. The proof is very similar in spirit to the rank-one case proof in [DST13].

Theorem 2.6. *Consider a multiplier matrix in the form*

$$(2.30) \quad \mathbf{R}(z) = \mathbf{I} + \frac{z_\alpha - z_\beta}{z - z_\alpha} \mathcal{P}, \quad \text{where } \mathcal{P} = \mathcal{P}_1 + \mathcal{P}_2 = \tilde{\mathcal{P}}_1 + \tilde{\mathcal{P}}_2,$$

and $\mathcal{P}_i, \tilde{\mathcal{P}}_i$ are given by (2.21–2.22). Then \mathcal{P} satisfies the constraints (2.6) and so defines a Schlesinger transformation. This Schlesinger transformation has the type $\left\{ \begin{smallmatrix} \alpha & \beta \\ 1 & 1 \\ 2 & 2 \end{smallmatrix} \right\}$ and the corresponding map

$$(\mathcal{B} \times \mathcal{C})_{\Theta} \rightarrow (\tilde{\mathcal{B}} \times \tilde{\mathcal{C}})_{\tilde{\Theta}}$$

is given by the following evolution equations, where c_i^j are again arbitrary non-zero constants.

(i) Transformation vectors:

$$(2.31) \quad \bar{\mathbf{b}}_{\alpha,1} = \frac{1}{c_\alpha^1} \mathbf{Q}_2 \mathbf{b}_{\beta,1}, \quad \bar{\mathbf{b}}_{\alpha,2} = \frac{1}{c_\alpha^2} \mathbf{Q}_1 \mathbf{b}_{\beta,2}, \quad \bar{\mathbf{c}}_\beta^{1\dagger} = c_\beta^1 \mathbf{c}_\alpha^{1\dagger} \mathbf{Q}_2, \quad \bar{\mathbf{c}}_\beta^{2\dagger} = c_\beta^2 \mathbf{c}_\alpha^{2\dagger} \mathbf{Q}_1.$$

(ii) *Generic indices:*

$$(2.32) \quad \bar{\mathbf{b}}_{i,j} = \frac{1}{c_i^j} \mathbf{R}(z_i) \mathbf{b}_{i,j}, \quad (i \neq \alpha \text{ and if } i = \beta, j > 2);$$

$$(2.33) \quad \bar{\mathbf{c}}_i^{j\dagger} = c_i^j \mathbf{c}_i^{j\dagger} \mathbf{R}^{-1}(z_i), \quad (i \neq \beta \text{ and if } i = \alpha, j > 2).$$

(iii) *Special indices (here $k = 1, 2$, $k' = 3 - k$, and $j > 2$):*

$$(2.34) \quad \bar{\mathbf{b}}_{\alpha,j} = \frac{1}{c_\alpha^j} \left(\mathbf{I} - \left(\frac{\mathcal{P}_1}{\theta_\alpha^1 - \theta_\alpha^j - 1} + \frac{\mathcal{P}_2}{\theta_\alpha^2 - \theta_\alpha^j - 1} \right) \left(\sum_{i \neq \alpha} \frac{z_\beta - z_\alpha}{z_i - z_\alpha} \mathbf{A}_i \right) \right) \mathbf{b}_{\alpha,j};$$

$$(2.35) \quad \bar{\mathbf{c}}_\beta^{j\dagger} = c_\beta^j \mathbf{c}_\beta^{j\dagger} \left(\mathbf{I} - \left(\sum_{i \neq \beta} \frac{z_\alpha - z_\beta}{z_i - z_\beta} \mathbf{A}_i \right) \left(\frac{\tilde{\mathcal{P}}_1}{\theta_\beta^1 - \theta_\beta^j + 1} + \frac{\tilde{\mathcal{P}}_2}{\theta_\beta^2 - \theta_\beta^j + 1} \right) \right);$$

$$(2.36) \quad \bar{\mathbf{b}}_{\beta,k} = \frac{1}{c_\beta^k} \left((\theta_\beta^k + 1) \mathbf{I} + \mathcal{Q} \left(\mathbf{I} + \sum_{j>2} \frac{\mathbf{b}_{\beta,j} \mathbf{c}_\beta^{j\dagger}}{\theta_\beta^1 - \theta_\beta^j + 1} \right) \left(\sum_{i \neq \beta} \frac{z_\alpha - z_\beta}{z_i - z_\beta} \mathbf{A}_i \right) \right) \frac{\mathbf{b}_{\beta,k}}{\mathbf{c}_\alpha^{k\dagger} \mathbf{Q}_{k'} \mathbf{b}_{\beta,k}};$$

$$(2.37) \quad \bar{\mathbf{c}}_\alpha^{k\dagger} = c_\alpha^k \frac{\mathbf{c}_\alpha^{k\dagger}}{\mathbf{c}_\alpha^{k\dagger} \mathbf{Q}_{k'} \mathbf{b}_{\beta,k}} \left((\theta_\alpha^k - 1) \mathbf{I} + \left(\sum_{i \neq \alpha} \frac{z_\beta - z_\alpha}{z_i - z_\alpha} \mathbf{A}_i \right) \left(\mathbf{I} + \sum_{j>2} \frac{\mathbf{b}_{\alpha,j} \mathbf{c}_\alpha^{j\dagger}}{\theta_\alpha^k - \theta_\alpha^j - 1} \right) \mathcal{Q} \right).$$

Proof. Of course the statement that \mathcal{P} defines an elementary Schlesinger transformation of the type $\begin{Bmatrix} \alpha & \beta \\ 1 & 1 \\ 2 & 2 \end{Bmatrix}$ follows from how we derived it, but it can also be seen directly. E.g., conditions (2.21–2.22) follow immediately from

$$(2.38) \quad \mathcal{P} \mathbf{A}_\alpha = (\theta_\alpha^1 \mathcal{P}_1 + \theta_\alpha^2 \mathcal{P}_2), \quad \mathbf{A}_\beta \mathcal{P} = \theta_\beta^1 \tilde{\mathcal{P}}_1 + \theta_\beta^2 \tilde{\mathcal{P}}_2,$$

and the fact that

$$\bar{\theta}_\alpha^i = \theta_\alpha^i - 1, \quad \bar{\theta}_\beta^i = \theta_\beta^i + 1 \quad \text{for } i = 1, 2 \quad \text{and} \quad \bar{\theta}_i^j = \theta_i^j \quad \text{otherwise}$$

can be seen, in particular, from our derivation of the evolution equations below.

To establish (i), we use (2.5):

$$\bar{\mathbf{A}}_\alpha \mathcal{P} = \mathcal{P} \mathbf{A}_\alpha - \mathcal{P} = (\theta_\alpha^1 - 1) \mathcal{P}_1 + (\theta_\alpha^2 - 1) \mathcal{P}_2.$$

Since $\mathcal{P}_1 \mathbf{Q}_2 \mathbf{b}_{\beta,1} = \mathbf{Q}_2 \mathbf{b}_{\beta,1}$ and $\mathcal{P}_2 \mathbf{Q}_2 \mathbf{b}_{\beta,1} = \mathbf{0}$, we see that $\bar{\theta}_\alpha^1 = \theta_\alpha^1 - 1$ and $\bar{\mathbf{b}}_{\alpha,1} \sim \mathbf{Q}_2 \mathbf{b}_{\beta,1}$, where \sim stands for ‘proportional’. Then $\bar{\mathbf{b}}_{\alpha,1} = \mathbf{Q}_2 \mathbf{b}_{\beta,1} / c_\alpha^1$, where c_α^1 is some non-zero proportionality constant. The other equations in this part are proved similarly. Note that the consequence of (i) is that we can write

$$(2.39) \quad \mathcal{P}_i = \frac{\bar{\mathbf{b}}_{\alpha,i} \mathbf{c}_\alpha^{i\dagger}}{\mathbf{c}_\alpha^{i\dagger} \bar{\mathbf{b}}_{\alpha,i}}, \quad \tilde{\mathcal{P}}_i = \frac{\mathbf{b}_{\beta,i} \bar{\mathbf{c}}_\beta^{i\dagger}}{\bar{\mathbf{c}}_\beta^{i\dagger} \mathbf{b}_{\beta,i}}, \quad \mathcal{P} = \frac{\bar{\mathbf{b}}_{\alpha,1} \mathbf{c}_\alpha^{1\dagger}}{\mathbf{c}_\alpha^{1\dagger} \bar{\mathbf{b}}_{\alpha,1}} + \frac{\bar{\mathbf{b}}_{\alpha,2} \mathbf{c}_\alpha^{2\dagger}}{\mathbf{c}_\alpha^{2\dagger} \bar{\mathbf{b}}_{\alpha,2}} = \frac{\mathbf{b}_{\beta,1} \bar{\mathbf{c}}_\beta^{1\dagger}}{\bar{\mathbf{c}}_\beta^{1\dagger} \mathbf{b}_{\beta,1}} + \frac{\mathbf{b}_{\beta,2} \bar{\mathbf{c}}_\beta^{2\dagger}}{\bar{\mathbf{c}}_\beta^{2\dagger} \mathbf{b}_{\beta,2}}.$$

For the generic case $i \neq \alpha, \beta$ in (ii) the proof is identical to the rank-one case. Since it is also short, we opted to include it to make the paper more self-contained. From (2.7) we see that

$$\bar{\mathbf{A}}_i \mathbf{R}(z_i) \mathbf{B}_i = \mathbf{R}(z_i) \mathbf{A}_i \mathbf{B}_i = \mathbf{R}(z_i) \mathbf{B}_i \boldsymbol{\Theta}_i,$$

and so $\bar{\boldsymbol{\Theta}}_i = \boldsymbol{\Theta}_i$ and $\bar{\mathbf{B}}_i \bar{\mathbf{D}}_i = \mathbf{R}(z_i) \mathbf{B}_i$, where $\bar{\mathbf{D}}_i = \text{diag}\{c_i^j\}$ is a diagonal matrix of non-zero proportionality constants. Similarly, $\bar{\mathbf{A}}_i \bar{\mathbf{C}}_i^\dagger = \mathbf{C}_i^\dagger \mathbf{R}^{-1}(z_i)$. The orthogonality condition $\bar{\mathbf{C}}_i^\dagger \bar{\mathbf{B}}_i = \boldsymbol{\Theta}_i$ implies that $\bar{\mathbf{A}}_i \bar{\mathbf{D}}_i = \mathbf{I}$, which gives (ii) for generic indices. For $i = \alpha$, from (2.4) we see that $\mathbf{C}_\alpha^\dagger \mathcal{Q} \bar{\mathbf{A}}_\alpha = \boldsymbol{\Theta}_\alpha \mathbf{C}_\alpha^\dagger \mathcal{Q}$, and so again

$\bar{\mathbf{A}}_\alpha \bar{\mathbf{C}}_\alpha^\dagger = \mathbf{C}_\alpha^\dagger \mathcal{Q}$. However, since $\mathbf{c}_\alpha^{1\dagger} \mathcal{Q} = \mathbf{c}_\alpha^{2\dagger} \mathcal{Q} = \mathbf{0}$, $(\bar{\mathbf{A}}_\alpha)_1^1 = (\bar{\mathbf{A}}_\alpha)_2^2 = 0$ and we can not recover $\bar{\mathbf{c}}_\alpha^{1\dagger}$ and $\bar{\mathbf{c}}_\alpha^{2\dagger}$. The case $i = \beta$ is similar.

Finally, let us consider special indices. To find $\bar{\mathbf{b}}_{\alpha,j}$ for $j > 2$ start with (2.8) and (2.33):

$$\bar{\mathbf{A}}_\alpha = \bar{\mathbf{b}}_{\alpha,1} \bar{\mathbf{c}}_\alpha^1 + \bar{\mathbf{b}}_{\alpha,2} \bar{\mathbf{c}}_\alpha^2 + \sum_{j>2} \bar{\mathbf{b}}_{\alpha,j} (c_\alpha^j \mathcal{Q} c_\alpha^j) = \mathbf{A}_\alpha - \mathcal{Q} \mathbf{A}_\alpha \mathcal{P} + \sum_{i \neq \alpha} \left(\frac{z_\beta - z_\alpha}{z_i - z_\alpha} \right) \mathcal{P} \mathbf{A}_i \mathcal{Q}.$$

Multiplying on the right by $\mathbf{b}_{\alpha,j}$, using the orthogonality conditions and $\mathcal{P} \mathbf{b}_{\alpha,j} = \mathbf{0}$, $\mathcal{Q} \mathbf{b}_{\alpha,j} = \mathbf{b}_{\alpha,j}$, we get

$$(2.40) \quad \bar{\mathbf{b}}_{\alpha,1} (\bar{\mathbf{c}}_\alpha^{1\dagger} \mathbf{b}_{\alpha,j}) + \bar{\mathbf{b}}_{\alpha,2} (\bar{\mathbf{c}}_\alpha^{2\dagger} \mathbf{b}_{\alpha,j}) + c_\alpha^j \theta_\alpha^j \bar{\mathbf{b}}_{\alpha,j} = \theta_\alpha^j \mathbf{b}_{\alpha,j} + \sum_{i \neq \alpha} \left(\frac{z_\beta - z_\alpha}{z_i - z_\alpha} \right) \mathcal{P} \mathbf{A}_i \mathbf{b}_{\alpha,j}.$$

Now left-multiply by $\bar{\mathbf{c}}_\alpha^{1\dagger}$ and use expression (2.39) for \mathcal{P} and orthogonality conditions again to get

$$\bar{\theta}_\alpha^1 (\bar{\mathbf{c}}_\alpha^{1\dagger} \mathbf{b}_{\alpha,j}) = \theta_\alpha^j (\bar{\mathbf{c}}_\alpha^{1\dagger} \mathbf{b}_{\alpha,j}) + \bar{\theta}_\alpha^1 \frac{\mathbf{c}_\alpha^{1\dagger}}{\mathbf{c}_\alpha^{1\dagger} \bar{\mathbf{b}}_{\alpha,1}} \sum_{i \neq \alpha} \left(\frac{z_\beta - z_\alpha}{z_i - z_\alpha} \right) \mathbf{A}_i \mathbf{b}_{\alpha,j}.$$

This gives

$$\begin{aligned} (\bar{\mathbf{c}}_\alpha^{1\dagger} \mathbf{b}_{\alpha,j}) &= \frac{\bar{\theta}_\alpha^1}{\bar{\theta}_\alpha^1 - \theta_\alpha^j} \frac{\mathbf{c}_\alpha^{1\dagger}}{\mathbf{c}_\alpha^{1\dagger} \bar{\mathbf{b}}_{\alpha,1}} \sum_{i \neq \alpha} \left(\frac{z_\beta - z_\alpha}{z_i - z_\alpha} \right) \mathbf{A}_i \mathbf{b}_{\alpha,j}, \\ \bar{\mathbf{b}}_{\alpha,1} (\bar{\mathbf{c}}_\alpha^{1\dagger} \mathbf{b}_{\alpha,j}) &= \frac{\bar{\theta}_\alpha^1}{\bar{\theta}_\alpha^1 - \theta_\alpha^j} \mathcal{P}_1 \sum_{i \neq \alpha} \left(\frac{z_\beta - z_\alpha}{z_i - z_\alpha} \right) \mathbf{A}_i \mathbf{b}_{\alpha,j}. \end{aligned}$$

Repeating the same steps for $\bar{\mathbf{b}}_{\alpha,2} (\bar{\mathbf{c}}_\alpha^{2\dagger} \mathbf{b}_{\alpha,j})$, substituting the result into (2.40), solving for $\bar{\mathbf{b}}_{\alpha,j}$ and simplifying gives (2.34); (2.35) is proved in a similar fashion.

Finally, to get expressions for $\bar{\mathbf{b}}_{\beta,1}$ and $\bar{\mathbf{b}}_{\beta,2}$, use all of the previously obtained expressions to write

$$\begin{aligned} \bar{\mathbf{A}}_\beta &= \bar{\mathbf{b}}_{\beta,1} \bar{\mathbf{c}}_\beta^{1\dagger} + \bar{\mathbf{b}}_{\beta,2} \bar{\mathbf{c}}_\beta^{2\dagger} + \sum_{j>2} \bar{\mathbf{b}}_{\beta,j} \bar{\mathbf{c}}_\beta^{j\dagger} \\ &= c_\beta^1 \bar{\mathbf{b}}_{\beta,1} \mathbf{c}_\alpha^{1\dagger} \mathbf{Q}_2 + c_\beta^2 \bar{\mathbf{b}}_{\beta,2} \mathbf{c}_\alpha^{2\dagger} \mathbf{Q}_1 \\ &\quad + \sum_{j>2} \mathcal{Q} \mathbf{b}_{\beta,j} \mathbf{c}_\beta^{j\dagger} \left(\mathbf{I} - \left(\sum_{i \neq \beta} \frac{z_\alpha - z_\beta}{z_i - z_\beta} \mathbf{A}_i \right) \left(\frac{\tilde{\mathcal{P}}_1}{\theta_\beta^1 - \theta_\beta^j + 1} + \frac{\tilde{\mathcal{P}}_2}{\theta_\beta^2 - \theta_\beta^j + 1} \right) \right) \end{aligned}$$

which, in view of (2.9), also can be written as

$$= \mathbf{A}_\beta - \mathcal{P} \mathbf{A}_\beta \mathcal{Q} + \mathcal{P} + \sum_{i \neq \beta} \left(\frac{z_\alpha - z_\beta}{z_i - z_\beta} \right) \mathcal{Q} \mathbf{A}_i \mathcal{P}.$$

Multiplying on the right by $\mathbf{b}_{\beta,1}$ we get

$$\begin{aligned} \bar{\mathbf{A}}_\beta \mathbf{b}_{\beta,1} &= c_\beta^1 (\mathbf{c}_\alpha^{1\dagger} \mathbf{Q}_2 \mathbf{b}_{\beta,1}) \bar{\mathbf{b}}_{\beta,1} + \sum_{j>2} \mathcal{Q} \mathbf{b}_{\beta,j} \mathbf{c}_\beta^{j\dagger} \left(\mathbf{I} - \sum_{i \neq \beta} \frac{z_\alpha - z_\beta}{z_i - z_\beta} \frac{\mathbf{A}_i}{\theta_\beta^1 - \theta_\beta^j + 1} \right) \mathbf{b}_{\beta,1} \\ &= c_\beta^1 (\mathbf{c}_\alpha^{1\dagger} \mathbf{Q}_2 \mathbf{b}_{\beta,1}) \bar{\mathbf{b}}_{\beta,1} - \mathcal{Q} \left(\sum_{i \neq \beta} \frac{z_\alpha - z_\beta}{z_i - z_\beta} \mathbf{A}_i \right) \left(\sum_{j>2} \frac{\mathbf{b}_{\beta,j} \mathbf{c}_\beta^{j\dagger}}{\theta_\beta^1 - \theta_\beta^j + 1} \right) \mathbf{b}_{\beta,1} \\ &= \theta_\beta^1 \mathbf{b}_{\beta,1} + \mathbf{b}_{\beta,1} + \mathcal{Q} \left(\sum_{i \neq \beta} \frac{z_\alpha - z_\beta}{z_i - z_\beta} \mathbf{A}_i \right) \mathbf{b}_{\beta,1}. \end{aligned}$$

Solving for $\bar{\mathbf{b}}_{\beta,1}$ gives (2.36) for $k = 1$, and the expression for $\mathbf{b}_{\beta,2}$ is obtained by right-multiplying by $\mathbf{b}_{\beta,2}$ instead. Equations (2.37) are obtained along the same lines. \square

3. REDUCTIONS FROM SCHLESINGER TRANSFORMATIONS TO DIFFERENCE PAINLEVÉ EQUATIONS

In this section, which is the central section of the paper, we consider two examples of reductions from the Schlesinger dynamic on the decomposition space to difference Painlevé equations. First we consider Schlesinger transformations of a Fuchsian system of spectral type 111, 111, 111. Resulting difference Painlevé equation is of type $d-P(A_2^{(1)*})$ and has the symmetry group $E_6^{(1)}$. We have previously considered this example in [DST13], but the exposition there was very brief and it relied on a nontrivial observation on how to choose good coordinates parameterizing our Fuchsian system. Here we not only provide more details but also show how geometric considerations *lead us* to the appropriate coordinate choice. In the second example we consider Schlesinger transformations of a Fuchsian system of spectral type 22, 1111, 1111, which gives difference Painlevé equation of type $d-P(A_1^{(1)*})$ with the symmetry group $E_7^{(1)}$. This example is completely new and here, in addition to elementary Schlesinger transformations of rank one, we also, for the first time, consider elementary Schlesinger transformations of rank two — we need such transformations to represent the standard example of a difference Painlevé equation of type $d-P(A_1^{(1)*})$, as written in [GRO03], [Sak07], as a composition of elementary Schlesinger transformations.

3.1. Reduction to difference Painlevé equation of type $d-P(A_2^{(1)*})$ with the symmetry group $E_6^{(1)}$.

3.1.1. *Model Example.* For our model example of type $d-P(A_2^{(1)*})$ we take the equation that was first written by Grammaticos, Ramani, and Ohta, [GRO03], see also Murata [Mur04] and Sakai [Sak07]. Following Sakai's geometric approach, we view this equation as a birational map $\varphi : \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ with parameters b_1, \dots, b_8

$$(3.1) \quad \begin{pmatrix} b_1 & b_2 & b_3 & b_4 \\ b_5 & b_6 & b_7 & b_8 \end{pmatrix}; f, g \mapsto \begin{pmatrix} \bar{b}_1 & \bar{b}_2 & \bar{b}_3 & \bar{b}_4 \\ \bar{b}_5 & \bar{b}_6 & \bar{b}_7 & \bar{b}_8 \end{pmatrix}; \bar{f}, \bar{g},$$

where $\bar{b}_1 = b_1$, $\bar{b}_2 = b_2$, $\bar{b}_3 = b_3$, $\bar{b}_4 = b_4$, $\bar{b}_5 = b_5 + \delta$, $\bar{b}_6 = b_6 + \delta$, $\bar{b}_7 = b_7 - \delta$, $\bar{b}_8 = b_8 - \delta$, $\delta = b_1 + \dots + b_8$, and \bar{f} and \bar{g} are given by the equation

$$(3.2) \quad \begin{cases} (f+g)(\bar{f}+g) = \frac{(g+b_1)(g+b_2)(g+b_3)(g+b_4)}{(g-b_5)(g-b_6)} \\ (\bar{f}+g)(\bar{f}+\bar{g}) = \frac{(\bar{f}-b_1)(\bar{f}-b_2)(\bar{f}-b_3)(\bar{f}-b_4)}{(\bar{f}+b_7-\delta)(\bar{f}+b_8-\delta)} \end{cases}.$$

This map has the following eight indeterminate points:

$$\begin{aligned} & p_1(b_1, -b_1), \quad p_3(b_3, -b_3), \quad p_5(\infty, b_5), \quad p_7(-b_7, \infty), \\ & p_2(b_2, -b_2), \quad p_4(b_4, -b_4), \quad p_6(\infty, b_6), \quad p_8(-b_8, \infty), \end{aligned}$$

resolving which by the blow-up procedure then gives us a rational surface $\mathcal{X}_{\mathbf{b}}$, known as the *Okamoto space of initial conditions* for this difference Painlevé equation, that is described by the blow-up diagram on Figure 2.

The Picard lattice of $\mathcal{X}_{\mathbf{b}}$ is generated by the total transforms H_f and H_g of the coordinate lines and the classes of the exceptional divisors E_i ,

$$\text{Pic}(\mathcal{X}) = \mathbb{Z}H_f \oplus \mathbb{Z}H_g \oplus \bigoplus_{i=1}^8 \mathbb{Z}E_i.$$

The anti-canonical divisor $-K_{\mathcal{X}} = 2H_f + 2H_g - \sum_{i=1}^8 E_i$ uniquely decomposes as a positive linear combination of -2 -curves D_i , $-K_{\mathcal{X}} = D_0 + D_1 + D_2$, where the irreducible components D_i , in bold on Figure 2, are given by

$$D_0 = H_f + H_g - E_1 - E_2 - E_3 - E_4, \quad D_1 = H_f - E_5 - E_6, \quad D_2 = H_g - E_7 - E_8.$$

The configuration of components D_i is described by the Dynkin diagram of type $A_2^{(1)*}$ (with nodes corresponding to classes of self-intersection -2 and edges connecting classes of intersection index 1). To this diagram correspond two different types of surfaces, the generic one corresponds to the multiplicative system of type $A_2^{(1)}$, and the degenerate configuration, where all three components D_i intersect at one point, corresponds to the additive system denoted by $A_2^{(1)*}$, which is clearly our case, see Figure 3.

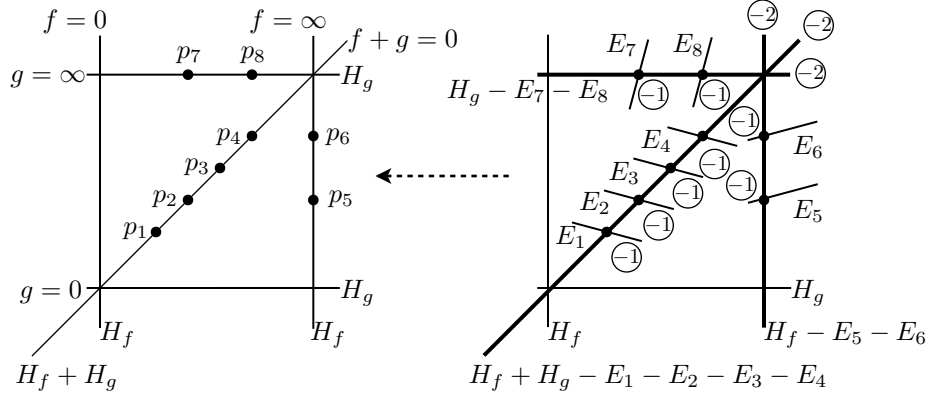


FIGURE 2. Okamoto surface $\mathcal{X}_{\mathbf{b}}$ for the model form of $d-P(A_2^{(1)*})$.

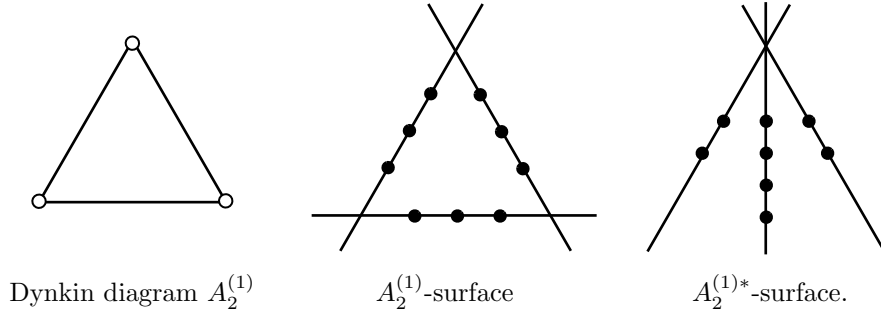


FIGURE 3. Configurations of type $A_2^{(1)}$

Components D_i of $-K_{\mathcal{X}}$ span the sub-lattice $R = \text{Span}_{\mathbb{Z}}\{D_1, D_2, D_3\}$, and its orthogonal complement R^{\perp} is called the *symmetry sub-lattice*. In our case, it is easy to see that $R^{\perp} = \text{Span}_{\mathbb{Z}}\{\alpha_0, \dots, \alpha_6\}$ is of type $E_6^{(1)}$, see Figure 4.

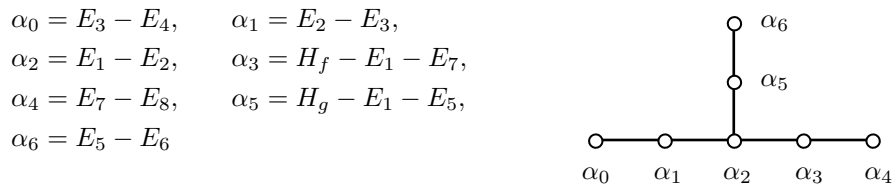


FIGURE 4. Symmetry sub-lattice for $d-P(\tilde{A}_2^*)$

Finally, we compute the action of φ_* on $\text{Pic}(\mathcal{X})$ to be

$$(3.3) \quad \begin{aligned} H_f &\mapsto 6H_f + 3H_g - 2E_1 - 2E_2 - 2E_3 - 2E_4 - E_5 - E_6 - 3E_7 - 3E_8, \\ H_g &\mapsto 3H_f + H_g - E_1 - E_2 - E_3 - E_4 - E_7 - E_8, \\ E_1 &\mapsto 2H_f + H_g - E_2 - E_3 - E_4 - E_7 - E_8, \\ E_2 &\mapsto 2H_f + H_g - E_1 - E_3 - E_4 - E_7 - E_8, \\ E_3 &\mapsto 2H_f + H_g - E_1 - E_2 - E_4 - E_7 - E_8, \\ E_4 &\mapsto 2H_f + H_g - E_1 - E_2 - E_3 - E_7 - E_8, \\ E_5 &\mapsto 3H_f + H_g - E_1 - E_2 - E_3 - E_4 - E_6 - E_7 - E_8, \\ E_6 &\mapsto 3H_f + H_g - E_1 - E_2 - E_3 - E_4 - E_5 - E_7 - E_8, \\ E_7 &\mapsto H_f - E_8, \\ E_8 &\mapsto H_f - E_7, \end{aligned}$$

and so the induced action φ_* on the sub-lattice R^\perp is given by the following *translation*:

$$(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6) \mapsto (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6) + (0, 0, 0, 1, 0, -1, 0)(-K_{\mathcal{X}}),$$

as well as the permutation $(D_0 D_1 D_2)$ of the irreducible components of $-K_{\mathcal{X}}$. We now want to compare this standard picture with the one that is obtained from Schlesinger transformations.

3.1.2. Schlesinger Transformations. Consider a 3×3 Fuchsian system of the spectral type 111, 111, 111. This system has three poles and it is convenient to assume that one of them is at $z_3 = \infty$, since our elementary Schlesinger transformations preserve \mathbf{A}_∞ . Also, in view of scalar gauge transformations we can assume that $\text{rank}(\mathbf{A}_i) = 2$ at finite poles (and using Möbius transformation preserving $z = \infty$, we can in principle map those poles to $z_1 = 0$ and $z_2 = 1$). Thus,

$$\mathbf{A}_i = \mathbf{B}_i \mathbf{C}_i^\dagger = \begin{bmatrix} \mathbf{b}_{i,1} & \mathbf{b}_{i,2} \end{bmatrix} \begin{bmatrix} \mathbf{c}_i^{1\dagger} \\ \mathbf{c}_i^{2\dagger} \end{bmatrix}, \quad i = 1, 2.$$

So the *Riemann scheme* and the *Fuchs relation* for our system are

$$\left\{ \begin{array}{ccc} z_1 & z_2 & z_3 \\ \theta_1^1 & \theta_2^1 & \theta_3^1 \\ \theta_1^2 & \theta_2^2 & \theta_3^2 \\ 0 & 0 & \theta_3^3 \end{array} \right\}, \quad \theta_1^1 + \theta_1^2 + \theta_2^1 + \theta_2^2 + \sum_{j=1}^3 \theta_3^j = 0.$$

This example does not have any continuous deformation parameters but it admits non-trivial Schlesinger transformation. Consider an elementary Schlesinger transformation $\left\{ \begin{smallmatrix} 1 & 2 \\ 1 & 1 \end{smallmatrix} \right\}$ that changes $\bar{\theta}_1^1 = \theta_1^1 - 1$, $\bar{\theta}_2^1 = \theta_2^1 + 1$, and fixes the remaining characteristic indices. The projector matrices for this map are

$$\mathbf{P} = \frac{\mathbf{b}_{2,1} \mathbf{c}_1^{1\dagger}}{\mathbf{c}_1^{1\dagger} \mathbf{b}_{2,1}}, \quad \mathbf{Q} = \mathbf{I} - \mathbf{P},$$

and the evolution equations (2.13–2.19) take the form

$$\begin{aligned} \bar{\mathbf{b}}_{1,1} &= \frac{1}{c_1^1} \mathbf{b}_{2,1}, & \bar{\mathbf{b}}_{1,2} &= \frac{1}{c_1^2} \left(\mathbf{I} - \frac{\mathbf{P} \mathbf{A}_2}{\theta_1^1 - \theta_1^2 - 1} \right) \mathbf{b}_{1,2}, \\ \bar{\mathbf{b}}_{2,2} &= \frac{1}{c_2^2} \mathbf{Q} \mathbf{b}_{2,2}, & \bar{\mathbf{b}}_{2,1} &= \frac{1}{c_2^1} \left((\theta_2^1 + 1) \mathbf{I} + \mathbf{Q} \left(\mathbf{I} + \frac{\mathbf{b}_{2,2} \mathbf{c}_2^{2\dagger}}{\theta_2^1 - \theta_2^2 + 1} \right) \mathbf{A}_1 \right) \frac{\mathbf{b}_{2,1}}{\mathbf{c}_1^{1\dagger} \mathbf{b}_{2,1}}, \\ \bar{\mathbf{c}}_1^{2\dagger} &= c_1^2 \mathbf{c}_1^{2\dagger} \mathbf{Q}, & \bar{\mathbf{c}}_1^{1\dagger} &= c_1^1 \frac{\mathbf{c}_1^{1\dagger}}{\mathbf{c}_1^{1\dagger} \mathbf{b}_{2,1}} \left((\theta_1^1 - 1) \mathbf{I} + \mathbf{A}_2 \left(\mathbf{I} + \frac{\mathbf{b}_{1,2} \mathbf{c}_1^{2\dagger}}{\theta_1^1 - \theta_1^2 - 1} \right) \mathbf{Q} \right), \\ \mathbf{c}_2^{1\dagger} &= c_2^1 \mathbf{c}_1^{1\dagger}, & \mathbf{c}_2^{2\dagger} &= c_2^2 \mathbf{c}_2^{2\dagger} \left(\mathbf{I} - \frac{\mathbf{A}_1 \mathbf{P}}{\theta_2^1 - \theta_2^2 + 1} \right), \end{aligned}$$

where c_i^j are arbitrary non-zero constants (corresponding to trivial gauge transformations).

We now explicitly show that the space of accessory parameters for Fuchsian systems of this type is two-dimensional by using various gauge transformations to put vectors $\mathbf{b}_{i,j}$ and $\mathbf{c}_i^{j\dagger}$ in some normal form, and then introduce a coordinate system on this phase space. First, assuming that we are in a generic situation, we use a global similarity transformation to map the vectors $\mathbf{b}_{1,1}$, $\mathbf{b}_{1,2}$, and $\mathbf{b}_{2,1}$ to the standard basis, and then use trivial gauge transformations (i.e., choose appropriate constants c_i^j) to make all components of $\mathbf{b}_{2,2}$ equal to 1. Then the orthogonality conditions $\mathbf{C}_i^\dagger \mathbf{B}_i = \Theta_i$ give us the following parameterization:

$$\mathbf{B}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{C}_1^\dagger = \begin{bmatrix} \theta_1^1 & 0 & \alpha \\ 0 & \theta_1^2 & \beta \end{bmatrix}, \quad \mathbf{B}_2 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad \mathbf{C}_2^\dagger = \begin{bmatrix} -x - \theta_2^1 & x & \theta_2^1 \\ \theta_2^2 - y & y & 0 \end{bmatrix}.$$

Here we choose x and y as our coordinates, and we can express $\alpha = \alpha(x, y)$ and $\beta = \beta(x, y)$ from the condition that the eigenvalues of $\mathbf{A}_\infty = -\mathbf{B}_1 \mathbf{C}_1^\dagger - \mathbf{B}_2 \mathbf{C}_2^\dagger$ are κ_1 , κ_2 , and κ_3 (the resulting expressions, although easy to obtain, are quite large and we omit them). We then get the following dynamic in the coordinates (x, y) :

$$\begin{cases} \bar{x} = \frac{\alpha - \beta}{\alpha(\theta_1^2 - \theta_1^1 + 1)} (\alpha(x + y) + \theta_1^1 y) \\ \bar{y} = \frac{\alpha - \beta}{\alpha(\theta_1^2 - \theta_1^1 + 1)} \left(\frac{\alpha(\alpha(x + y) + y(\theta_1^2 + 1))(\theta_1^1 - \theta_2^2 + 1)}{\alpha(\theta_1^2 + 1) - (\alpha - \beta)y} - \alpha(x + y) - \theta_1^1 y \right) \end{cases},$$

where we still need to substitute $\alpha = \alpha(x, y)$ and $\beta = \beta(x, y)$. So this map is quite complicated and it reflects the fact that our choice of the coordinates was rather arbitrary. To better understand the map we again go back to geometry.

The indeterminate points of the map $\psi : (x, y) \rightarrow (\bar{x}, \bar{y})$ are

$$\begin{aligned} p_1 & \left(\frac{(\theta_1^1 + \theta_2^1 + \theta_3^1)(\theta_1^2 + \theta_3^1)}{\theta_1^1 - \theta_2^1}, -\frac{(\theta_1^1 + \theta_2^2 + \theta_3^1)(\theta_1^2 + \theta_3^1)}{\theta_1^1 - \theta_2^1} \right), \quad p_4(0, 0), \\ p_2 & \left(\frac{(\theta_1^1 + \theta_2^1 + \theta_3^2)(\theta_1^2 + \theta_3^2)}{\theta_1^1 - \theta_2^1}, -\frac{(\theta_1^1 + \theta_2^2 + \theta_3^2)(\theta_1^2 + \theta_3^2)}{\theta_1^1 - \theta_2^1} \right), \quad p_5(-\theta_2^1, \theta_2^2), \\ p_3 & \left(\frac{(\theta_1^1 + \theta_2^1 + \theta_3^3)(\theta_1^2 + \theta_3^3)}{\theta_1^1 - \theta_2^1}, -\frac{(\theta_1^1 + \theta_2^2 + \theta_3^3)(\theta_1^2 + \theta_3^3)}{\theta_1^1 - \theta_2^2} \right), \end{aligned}$$

as well as the sequence of infinitely close points

$$\begin{aligned} p_6 \left(\frac{1}{x} = 0, \frac{1}{y} = 0 \right) & \leftarrow p_7 \left(\frac{1}{x} = 0, \frac{x}{y} = -1 \right) \\ & \leftarrow p_8 \left(\frac{1}{x} = 0, \frac{x}{y} = -1, \frac{x(x + y)}{y} = \frac{(\theta_1^2 + 1)(\theta_2^1 - \theta_2^2)}{\theta_1^2 - \theta_1^1} \right). \end{aligned}$$

Note also that the points p_1, \dots, p_6 (and, after blowing up, the point p_7 as well) all lie on a $(2, 2)$ -curve Q given by the equation

$$(3.4) \quad (\theta_1^1 - \theta_1^2)(x + y)(x + y + \theta_2^1 - \theta_2^2) + (\theta_1^1 - \theta_2^2)(\theta_2^2 x + \theta_2^1 y) = 0.$$

Resolving indeterminate points of this map using blow-ups gives us the Okamoto surface \mathcal{X}_θ pictured on Figure 5. We can immediately see that in this case

$$-K_{\mathcal{X}_\theta} = (2H_x + 2H_y - F_1 - F_2 - F_3 - F_4 - F_5 - 2F_6 - F_7) + (F_6 - F_7) + (F_7 - F_8),$$

where F_i stand for classes of exceptional divisors, and, since all of the -2 -curves intersect at one point, \mathcal{X} indeed has the type $A_2^{(1)*}$. Unfortunately, two of the three irreducible components of $-K_{\mathcal{X}}$ are now completely in the blow-up region. This makes identification with the standard example more difficult since we have to go through a sequence of coordinate charts to do the computation. A better approach is to use \mathbb{P}^2 compactification of \mathbb{C}^2 (recall that in this case, according to general theory, we expect to have nine blow-up points instead of eight).

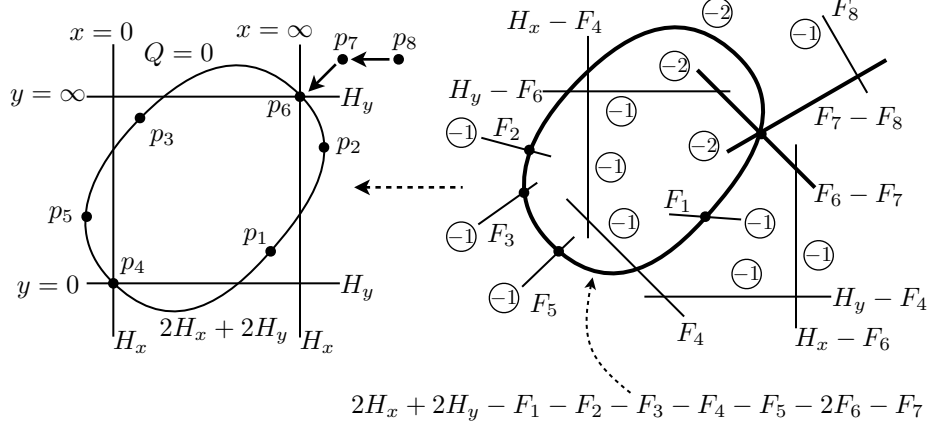


FIGURE 5. Okamoto surface \mathcal{X}_θ for the Schlesinger transformations reduction to $d-P(A_2^{(1)*})$ using $\mathbb{P}^1 \times \mathbb{P}^1$ compactification of \mathbb{C}^2 .

In this compactification we still have the same finite points that, in the homogeneous coordinates, are

$$\begin{aligned}
 p_1 & \left(\frac{(\theta_1^1 + \theta_2^1 + \theta_3^1)(\theta_1^2 + \theta_3^1)}{\theta_1^1 - \theta_1^2} : -\frac{(\theta_1^1 + \theta_2^2 + \theta_3^1)(\theta_1^2 + \theta_3^1)}{\theta_1^1 - \theta_1^2} : 1 \right), \quad p_4(0 : 0 : 1), \\
 p_4 & \left(\frac{(\theta_1^1 + \theta_2^1 + \theta_3^2)(\theta_1^2 + \theta_3^2)}{\theta_1^1 - \theta_1^2} : -\frac{(\theta_1^1 + \theta_2^2 + \theta_3^2)(\theta_1^2 + \theta_3^2)}{\theta_1^1 - \theta_1^2} : 1 \right), \quad p_5(-\theta_2^1 : \theta_2^2 : 1), \\
 p_3 & \left(\frac{(\theta_1^1 + \theta_2^1 + \theta_3^3)(\theta_1^2 + \theta_3^3)}{\theta_1^1 - \theta_1^2} : -\frac{(\theta_1^1 + \theta_2^2 + \theta_3^3)(\theta_1^2 + \theta_3^3)}{\theta_1^1 - \theta_1^2} : 1 \right).
 \end{aligned}$$

There are also three more points on the line at infinity, and one infinitely close point p_9 :

$$p_6(1 : -1 : 0) \leftarrow p_9 \left(0, \frac{\theta_1^1 - \theta_1^2}{(\theta_2^1 - \theta_2^2)(\theta_1^2 + 1)} \right), \quad p_7(0 : 1 : 0), \quad p_8(1 : 0 : 0),$$

where coordinates of p_9 are w.r.t the coordinate system $u = \frac{X+Y}{X}$, $v = \frac{Z}{X+Y}$ in the chart $X \neq 0$. Points p_1, \dots, p_6 lie on the projectivization of the $(2, 2)$ -curve Q whose homogeneous equation in \mathbb{P}^2 is

$$(3.5) \quad (\theta_1^1 - \theta_1^2)(X + Y)(X + Y + (\theta_1^1 - \theta_2^2)Z) + (\theta_2^1 - \theta_2^2)(\theta_2^2 X + \theta_2^1 Y)Z = 0.$$

The resulting blow-up diagram is depicted on Figure 6.

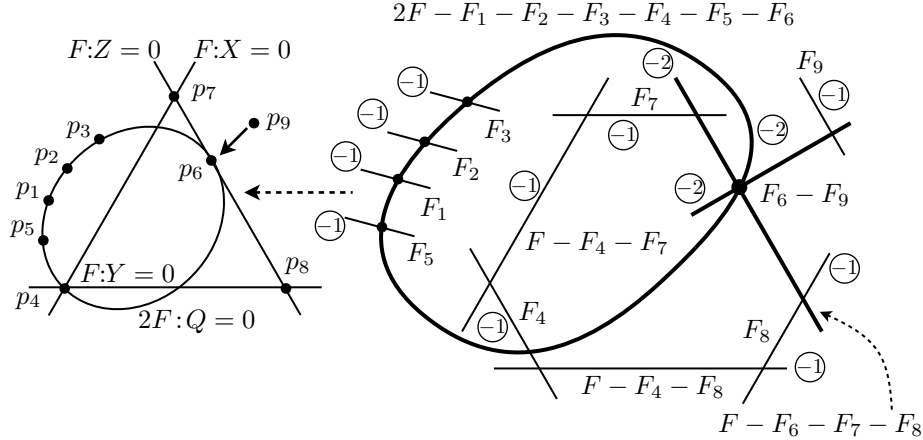


FIGURE 6. Okamoto surface \mathcal{X}_θ for the Schlesinger transformations reduction to $d-P(A_2^{(1)*})$ using \mathbb{P}^2 compactification of \mathbb{C}^2 .

As before, we see that the anti-canonical divisor $-K_{\mathcal{X}}$ uniquely decomposes as a positive linear combination of -2 -curves D_i ,

$$-K_{\mathcal{X}} = 3F - \sum_{i=1}^9 F_i = D_0 + D_1 + D_2,$$

where

$$D_0 = 2F - F_1 - F_2 - F_3 - F_4 - F_5 - F_6, \quad D_1 = F - F_6 - F_7 - F_8, \quad D_2 = F_6 - F_9.$$

The configuration of components D_i is again described by the Dynkin diagram of type $A_2^{(1)}$, and since all three -2 -curves intersect at one point, this is a surface of type $A_2^{(1)*}$. To compare this dynamic with the model example considered earlier we need to find an explicit isomorphism between the corresponding Okamoto surfaces, choose the same bases in the Picard lattice, and then compute the translation directions in the symmetry sub-lattice. This is what we do next.

3.1.3. Reduction to the standard form. To match the surface \mathcal{X}_{θ} described by the blow-up diagram on Figure 6 with the surface $\mathcal{X}_{\mathbf{b}}$ described by diagram on Figure 2, we look for the blow-down structure describing $\mathcal{X}_{\mathbf{b}}$ in $\text{Pic}(\mathcal{X}_{\theta})$, i.e., we look for rational classes $\mathcal{H}_f, \mathcal{H}_g, \mathcal{E}_1, \dots, \mathcal{E}_8$ in $\text{Pic}(\mathcal{X}_{\theta})$ such that

$$\mathcal{H}_f \bullet \mathcal{H}_g = 1, \quad \mathcal{E}_i^2 = -1, \quad \mathcal{H}_f^2 = \mathcal{H}_g^2 = \mathcal{H}_f \bullet \mathcal{E}_i = \mathcal{H}_g \bullet \mathcal{E}_i = \mathcal{E}_i \bullet \mathcal{E}_j = 0, \quad 1 \leq i \neq j \leq 8,$$

and the resulting configuration matches diagram on Figure 2. By the (virtual) genus formula $g(C) = (C^2 + K_{\mathcal{X}} \bullet C)/2 + 1$, we see that we should look for classes of rational curves of self-intersection zero among $F - F_i$ and for classes of rational curves of self-intersection -1 among F_i or $F - F_i - F_j$.

Comparing the -2 -curves on both diagrams,

$$\begin{aligned} D_0 &= 2F - F_1 - F_2 - F_3 - F_4 - F_5 - F_6 = \mathcal{H}_f + \mathcal{H}_g - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3 - \mathcal{E}_4, \\ D_1 &= F - F_6 - F_7 - F_8 = \mathcal{H}_f - \mathcal{E}_5 - \mathcal{E}_6, \\ D_2 &= F_6 - F_9 = \mathcal{H}_g - \mathcal{E}_7 - \mathcal{E}_8, \end{aligned}$$

we see that it makes sense to choose $\mathcal{E}_i = F_i$ for $i = 1, \dots, 4$. Then $\mathcal{H}_f + \mathcal{H}_g = F - F_5 - F_6$, and looking at D_1 we put $\mathcal{H}_f = F - F_6$, $\mathcal{E}_5 = F_7$, $\mathcal{E}_6 = F_8$. This then requires that $\mathcal{H}_g = F - F_5$, and looking at D_2 we get $\mathcal{E}_7 + \mathcal{E}_8 = F - F_5 - F_6 + F_9$. We put $\mathcal{E}_7 = F - F_5 - F_6$ (to ensure that $\mathcal{H}_f \bullet \mathcal{E}_7 = \mathcal{H}_g \bullet \mathcal{E}_7 = 0$), and then $\mathcal{E}_8 = F_9$. To summarize, we get the following identification, which clearly satisfies all of the required conditions

$$\begin{aligned} \mathcal{H}_f &= F - F_6, \quad \mathcal{E}_1 = F_1, \quad \mathcal{E}_3 = F_3, \quad \mathcal{E}_5 = F_7, \quad \mathcal{E}_7 = F - F_5 - F_6, \\ \mathcal{H}_g &= F - F_5, \quad \mathcal{E}_2 = F_2, \quad \mathcal{E}_4 = F_4, \quad \mathcal{E}_6 = F_8, \quad \mathcal{E}_8 = F_9. \end{aligned}$$

To complete the correspondence it remains to define the base coordinates f and g of the linear systems $|\mathcal{H}_f|$ and $|\mathcal{H}_g|$ that will map the exceptional fibers of the divisors \mathcal{E}_i to the points π_i such that π_5 and π_6 are on the line $f = \infty$, π_7 and π_8 are on $g = \infty$, and π_1, \dots, π_4 are on the line $f + g = 0$. Since the pencil $|\mathcal{H}_f|$ consists of all curves on \mathbb{P}^2 passing through $p_6(1 : -1 : 0)$,

$$|\mathcal{H}_f| = |F - F_6| = \{aX + bY + cZ = 0 \mid a - b = 0\} = \{a(X + Y) + cZ = 0\},$$

we can define the projective base coordinate as $f_1 = [X + Y : Z]$. Similarly,

$$\begin{aligned} |\mathcal{H}_g| &= |F - F_5| = \{aX + bY + cZ = 0 \mid -\theta_2^1 a + \theta_2^2 b + c = 0\} \\ &= \{a(X + \theta_2^1 Z) + b(Y - \theta_2^2 Z) = 0\}, \end{aligned}$$

and $g_1 = [X + \theta_2^1 Z : Y - \theta_2^2 Z]$. Then

$$\begin{aligned} f_1(\pi_5) &= f_1(p_7) = [-1 : 0], & g_1(\pi_7) &= g_1(p_6) = [1 : -1], \\ f_1(\pi_6) &= f_1(p_8) = [1 : 0], & g_1(\pi_8) &= g_1(p_6) = [1 : -1]. \end{aligned}$$

In order to have $g_1(\pi_7) = g_1(\pi_8) = \infty$ we first make an affine change of coordinates $\tilde{g}_1 = [(X + \theta_2^1 Z) + (Y - \theta_2^2 Z) : Y - \theta_2^2 Z]$ to get $\tilde{g}_1(\pi_7) = \tilde{g}_1(\pi_8) = [0 : -1]$ and then put

$$f_2 = \frac{X + Y}{Z}, \quad g_2 = \frac{Y - \theta_2^2 Z}{(X + Y) + (\theta_2^1 - \theta_2^2)Z}.$$

Equation (3.5) of the curve Q in these coordinates becomes

$$(3.6) \quad Z^2(f + \theta_2^1 - \theta_2^2)((\theta_1^1 - \theta_1^2)f_2 + (\theta_2^1 - \theta_2^2)((\theta_2^1 - \theta_2^2)g_2 + \theta_2^2)) = 0,$$

and the points π_1, \dots, π_4 lie on the line $(\theta_1^1 - \theta_1^2)f_2 + (\theta_2^1 - \theta_2^2)((\theta_2^1 - \theta_2^2)g_2 + \theta_2^2) = 0$. Thus, if we finally put

$$(3.7) \quad f = \frac{(\theta_1^1 - \theta_1^2)}{(\theta_2^1 - \theta_2^2)} f_2 = \frac{(\theta_1^1 - \theta_1^2)(X + Y)}{(\theta_2^1 - \theta_2^2)Z} = \frac{(\theta_1^1 - \theta_1^2)(x + y)}{(\theta_2^1 - \theta_2^2)},$$

$$(3.8) \quad g = (\theta_2^1 - \theta_2^2)g_2 + \theta_2^2 = \frac{\theta_2^2 X + \theta_2^1 Y}{(X + Y) + (\theta_2^1 - \theta_2^2)Z} = \frac{\theta_2^2 x + \theta_2^1 y}{(x + y) + (\theta_2^1 - \theta_2^2)},$$

points π_1, \dots, π_4 will be on the line $f + g = 0$, points π_5 and π_6 will be on the line $f = \infty$, and points π_7 and π_8 will be on the line $g = \infty$, as requires. Specifically, we get

$$\begin{aligned} \pi_1(\theta_1^2 + \theta_3^1, -\theta_1^2 - \theta_3^1), \quad \pi_3(\theta_1^2 + \theta_3^3, -\theta_1^2 - \theta_3^3), \quad \pi_5(\infty, \theta_2^1), \quad \pi_7(\theta_1^2 - \theta_1^1, \infty), \\ \pi_2(\theta_1^2 + \theta_3^2, -\theta_1^2 - \theta_3^2), \quad \pi_4(0, 0), \quad \pi_6(\infty, \theta_2^2), \quad \pi_8(\theta_1^2 + 1, \infty). \end{aligned}$$

Thus, we immediately get the identification between the parameters in the Riemann scheme of our Fuchsian system and the parameters b_i in the model equation:

$$\begin{aligned} b_1 = \theta_1^2 + \theta_3^1, \quad b_3 = \theta_1^2 + \theta_3^3, \quad b_5 = \theta_2^1, \quad b_7 = \theta_1^1 - \theta_1^2, \\ b_2 = \theta_1^2 + \theta_3^2, \quad b_4 = 0, \quad b_6 = \theta_2^2, \quad b_8 = -\theta_1^2 - 1. \end{aligned}$$

This, in turn, allows us to see the effect of the standard Painlevé dynamic on the Riemann scheme. Indeed, $\delta = b_1 + \dots + b_8 = -1$, and, for example, $\bar{k}_1 = \bar{b}_1 + \bar{b}_8 + 1 = b_1 + b_8 - \delta + 1 = k_1 + 1$, and so on. So for the model equation we get

$$\left\{ \begin{array}{ccc} z_1 & z_2 & z_3 \\ \theta_1^1 & \theta_2^1 & \theta_3^1 \\ \theta_1^2 & \theta_2^2 & \theta_3^2 \\ 0 & 0 & \theta_3^3 \end{array} \right\} \xrightarrow{\text{d-}P(A_2^{(1)*})} \left\{ \begin{array}{ccc} z_1 & z_2 & z_3 \\ \theta_1^1 & \theta_2^1 - 1 & \theta_3^1 + 1 \\ \theta_1^2 - 1 & \theta_2^2 - 1 & \theta_3^2 + 1 \\ 0 & 0 & \theta_3^3 + 1 \end{array} \right\},$$

whereas our elementary Schlesinger transformation acts as

$$\left\{ \begin{array}{ccc} z_1 & z_2 & z_3 \\ \theta_1^1 & \theta_2^1 & \theta_3^1 \\ \theta_1^2 & \theta_2^2 & \theta_3^2 \\ 0 & 0 & \theta_3^3 \end{array} \right\} \xrightarrow{\left\{ \begin{smallmatrix} 1 & 2 \\ 1 & 1 \end{smallmatrix} \right\}} \left\{ \begin{array}{ccc} z_1 & z_2 & z_3 \\ \theta_1^1 - 1 & \theta_2^1 + 1 & \theta_3^1 \\ \theta_1^2 & \theta_2^2 & \theta_3^2 \\ 0 & 0 & \theta_3^3 \end{array} \right\}.$$

Thus, these two transformations correspond to the different translation directions in the symmetry root sub-lattice of the surface \tilde{X} and so are not equivalent. Indeed, we compute the action of ψ_* of an elementary Schlesinger transformation $\left\{ \begin{smallmatrix} 1 & 2 \\ 1 & 1 \end{smallmatrix} \right\}$ on the classes \mathcal{H}_f , \mathcal{H}_g , and \mathcal{E}_i to be

$$\begin{aligned} \mathcal{H}_f &\mapsto 2\mathcal{H}_f + 3\mathcal{H}_g - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3 - \mathcal{E}_4 - 2\mathcal{E}_5 - 2\mathcal{E}_8, \\ \mathcal{H}_g &\mapsto 3\mathcal{H}_f + 5\mathcal{H}_g - 2\mathcal{E}_1 - 2\mathcal{E}_2 - 2\mathcal{E}_3 - 2\mathcal{E}_4 - 3\mathcal{E}_5 - \mathcal{E}_6 - 2\mathcal{E}_8, \\ \mathcal{E}_1 &\mapsto \mathcal{H}_f + 2\mathcal{H}_g - \mathcal{E}_2 - \mathcal{E}_3 - \mathcal{E}_4 - \mathcal{E}_5 - \mathcal{E}_8, \\ \mathcal{E}_2 &\mapsto \mathcal{H}_f + 2\mathcal{H}_g - \mathcal{E}_1 - \mathcal{E}_3 - \mathcal{E}_4 - \mathcal{E}_5 - \mathcal{E}_8, \\ \mathcal{E}_3 &\mapsto \mathcal{H}_f + 2\mathcal{H}_g - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_4 - \mathcal{E}_5 - \mathcal{E}_8, \\ \mathcal{E}_4 &\mapsto \mathcal{H}_f + 2\mathcal{H}_g - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3 - \mathcal{E}_5 - \mathcal{E}_8, \\ \mathcal{E}_5 &\mapsto \mathcal{E}_7, \\ \mathcal{E}_6 &\mapsto 2\mathcal{H}_f + 2\mathcal{H}_g - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3 - \mathcal{E}_4 - 2\mathcal{E}_5 - \mathcal{E}_8, \\ \mathcal{E}_7 &\mapsto 2\mathcal{H}_f + 3\mathcal{H}_g - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3 - \mathcal{E}_4 - 2\mathcal{E}_5 - \mathcal{E}_6 - 2\mathcal{E}_8, \\ \mathcal{E}_8 &\mapsto \mathcal{H}_g - \mathcal{E}_5, \end{aligned}$$

and compare with the standard dynamic φ_* given by (3.3) to see this explicitly:

$$\begin{aligned}\psi_* : (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6) &\mapsto (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6) + \\ &\quad (0, 0, 0, -1, 1, 1, -1) (-K_{\mathcal{X}}), \\ \varphi_* : (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6) &\mapsto (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6) + \\ &\quad (0, 0, 0, 1, 0, -1, 0) (-K_{\mathcal{X}}).\end{aligned}$$

It is possible to represent the standard Painlevé dynamic as a composition of two elementary Schlesinger transformations, combined with some automorphisms of our Fuchsian system. We first demonstrate this by looking at a sequence of actions on the Riemann scheme:

$$\begin{aligned}\left\{ \begin{array}{ccc} z_1 & z_2 & z_3 \\ \theta_1^1 & \theta_2^1 & \theta_3^1 \\ \theta_1^2 & \theta_2^2 & \theta_3^2 \\ 0 & 0 & \theta_3^3 \end{array} \right\} &\xrightarrow{\left\{ \begin{array}{cc} 2 & 1 \\ 1 & 1 \end{array} \right\}} \left\{ \begin{array}{ccc} z_1 & z_2 & z_3 \\ \theta_1^1 + 1 & \theta_2^1 - 1 & \theta_3^1 \\ \theta_1^2 & \theta_2^2 & \theta_3^2 \\ 0 & 0 & \theta_3^3 \end{array} \right\} \\ &\xrightarrow{\sigma_1(1,3)} \left\{ \begin{array}{ccc} z_1 & z_2 & z_3 \\ 0 & \theta_2^1 - 1 & \theta_3^1 \\ \theta_1^2 & \theta_2^2 & \theta_3^2 \\ \theta_1^1 + 1 & 0 & \theta_3^3 \end{array} \right\} \\ &\xrightarrow{\rho_1(-\theta_1^1 - 1)} \left\{ \begin{array}{ccc} z_1 & z_2 & z_3 \\ -\theta_1^1 - 1 & \theta_2^1 - 1 & \theta_3^1 + \theta_1^1 + 1 \\ \theta_1^2 - \theta_1^1 - 1 & \theta_2^2 & \theta_3^2 + \theta_1^1 + 1 \\ 0 & 0 & \theta_3^3 + \theta_1^1 + 1 \end{array} \right\} \\ &\xrightarrow{\left\{ \begin{array}{cc} 2 & 1 \\ 2 & 1 \end{array} \right\}} \left\{ \begin{array}{ccc} z_1 & z_2 & z_3 \\ -\theta_1^1 & \theta_2^1 - 1 & \theta_3^1 + \theta_1^1 + 1 \\ \theta_1^2 - \theta_1^1 - 1 & \theta_2^2 - 1 & \theta_3^2 + \theta_1^1 + 1 \\ 0 & 0 & \theta_3^3 + \theta_1^1 + 1 \end{array} \right\} \\ &\xrightarrow{\sigma_1(1,3)} \left\{ \begin{array}{ccc} z_1 & z_2 & z_3 \\ 0 & \theta_1^1 - 1 & \theta_3^1 + \theta_1^1 + 1 \\ \theta_1^2 - \theta_1^1 - 1 & \theta_2^2 - 1 & \theta_3^2 + \theta_1^1 + 1 \\ \theta_1^1 & 0 & \theta_3^3 + \theta_1^1 + 1 \end{array} \right\} \\ &\xrightarrow{\rho_1(\theta_1^1)} \left\{ \begin{array}{ccc} z_1 & z_2 & z_3 \\ \theta_1^1 & \theta_2^1 - 1 & \theta_3^1 + 1 \\ \theta_1^2 - 1 & \theta_2^2 - 1 & \theta_3^2 + 1 \\ 0 & 0 & \theta_3^3 + 1 \end{array} \right\}.\end{aligned}$$

Here $\left\{ \begin{array}{cc} 2 & 1 \\ 1 & 1 \end{array} \right\}$ and $\left\{ \begin{array}{cc} 2 & 1 \\ 2 & 1 \end{array} \right\}$ are the usual elementary Schlesinger transformations, the map $\rho_i(s) : \mathbf{A}(z) \mapsto (z - z_i)^s \mathbf{A}(z)$ is a scalar gauge transformation, and $\sigma_i(j, k)$ is a map that exchanges the j -th and the k -th eigenvectors (and eigenvalues) of \mathbf{A}_i . Note that, if the eigenvalues θ_i^j and θ_i^k are non-zero, this map is just a permutation on the decomposition space $\mathcal{B} \times \mathcal{C}$. And even though for the map $\sigma_1(1, 3)$ that we use above one of the eigenvectors has the eigenvalue zero, this map is still well-defined as a map on $\mathcal{B} \times \mathcal{C}$, since $\mathbf{b}_1^3 \in \text{Ker}(\mathbf{C}_1^\dagger)$ and $\mathbf{c}_1^{3\dagger} \in \text{Ker}(\mathbf{B}_1)$. In fact, if we combine $\sigma_1(1, 3)$ with $\rho_1(-\theta_1^1)$ to define a transformation $\Sigma_1(1, 3) = \rho_1(-\theta_1^1) \circ \sigma_1(1, 3)$,

$$\Sigma_1(1, 3) : \left\{ \begin{array}{ccc} z_1 & z_2 & z_3 \\ \theta_1^1 & \theta_2^1 & \theta_3^1 \\ \theta_1^2 & \theta_2^2 & \theta_3^2 \\ 0 & 0 & \theta_3^3 \end{array} \right\} \mapsto \left\{ \begin{array}{ccc} z_1 & z_2 & z_3 \\ -\theta_1^1 & \theta_2^1 & \theta_3^1 + \theta_1^1 \\ \theta_1^2 - \theta_1^1 & \theta_2^2 & \theta_3^2 + \theta_1^1 \\ 0 & 0 & \theta_3^3 + \theta_1^1 \end{array} \right\},$$

the action of $\Sigma_1(1, 3)$ on the decomposition space is explicitly given by

$$\Sigma_1(1, 3) : (\mathbf{b}_1^1, \mathbf{b}_1^2; \mathbf{c}_1^{1\dagger}, \mathbf{c}_1^{2\dagger}; \mathbf{b}_2^1, \mathbf{b}_2^2; \mathbf{c}_2^{1\dagger}, \mathbf{c}_2^{2\dagger}) \mapsto \left((\mathbf{c}_1^{1\dagger} \times \mathbf{c}_1^{2\dagger})^t, \mathbf{b}_1^2; \frac{-\theta_1^1(\mathbf{b}_1^1 \times \mathbf{b}_1^2)^t}{(\mathbf{c}_1^{1\dagger} \times \mathbf{c}_1^{2\dagger})(\mathbf{b}_1^1 \times \mathbf{b}_1^2)}, \mathbf{c}_1^{2\dagger}; \mathbf{b}_2^1, \mathbf{b}_2^2; \mathbf{c}_2^{1\dagger}, \mathbf{c}_2^{2\dagger} \right),$$

where \times is the usual cross-product and t denotes transposition. We also had to use the normalization condition $\mathbf{C}_1^\dagger \mathbf{B}_1 = \Theta_1$.

It is also possible to show, by a direct computation, that

$$\mathrm{d}\text{-}P(A_2^{(1)*}) = \Sigma_1(1, 3) \circ \left\{ \begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix} \right\} \circ \Sigma_1(1, 3) \circ \left\{ \begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix} \right\}$$

holds on the level of equations as well.

3.2. Reductions to difference Painlevé equation of type $\mathrm{d}\text{-}P(A_1^{(1)*})$ with the symmetry group $E_7^{(1)}$.

3.2.1. *Model Example.* For our model example of $\mathrm{d}\text{-}P(A_1^{(1)*})$ equation we take the equation that first was appeared in [GRO03] as an asymmetric $\mathrm{q}\text{-}P_{\mathrm{IV}}$ equation, and we use the variables as in Sakai's paper [Sak07]. We again consider $\mathrm{d}\text{-}P(A_1^{(1)*})$ to be a birational map $\varphi : \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ with parameters b, b_1, \dots, b_8 ,

$$\begin{pmatrix} b & b_1 & b_2 & b_3 & b_4; f, g \end{pmatrix} \mapsto \begin{pmatrix} \bar{b} & \bar{b}_1 & \bar{b}_2 & \bar{b}_3 & \bar{b}_4; \bar{f}, \bar{g} \end{pmatrix}, \quad \begin{aligned} \bar{b} &= b - \delta, \\ \bar{b}_i &= b_i, \quad i = 1, \dots, 8, \end{aligned}$$

$\delta = b_1 + \dots + b_8$, and \bar{f} and \bar{g} are given by the equations

$$(3.9) \quad \begin{cases} \frac{(g + f - 2b)(g + \bar{f} - b - \bar{b})}{(g + f)(g + \bar{f})} = \frac{\prod_{i=1}^4 (g - b + b_i)}{\prod_{i=5}^8 (g - b_i)} \\ \frac{(g + \bar{f} - b - \bar{b})(\bar{g} + \bar{f} - 2\bar{b})}{(g + \bar{f})(\bar{g} + \bar{f})} = \frac{\prod_{i=1}^4 (\bar{f} - \bar{b} - b_i)}{\prod_{i=5}^8 (\bar{f} + b_i)} \end{cases}.$$

It is convenient to introduce the notation

$$G_{14} = G_{14}(g) = \prod_{i=1}^4 (g - b + b_i), \quad G_{14}^i = G_{14}^i(g) = \prod_{j=1, j \neq i}^4 (g - b + b_j),$$

and similarly for G_{58} , F_{14} , and F_{58} . The maps φ is then given by the sequence $(f, g) \rightarrow (\bar{f}, g) \rightarrow (\bar{f}, \bar{g})$ described by the equations

$$(3.10) \quad \bar{f} = -\frac{(g - b - \bar{b})(f + g - 2b)G_{58} - g(f + g)G_{14}}{(f + g - 2b)G_{58} - (f + g)G_{14}},$$

$$(3.11) \quad \bar{g} = -\frac{(\bar{f} - 2\bar{b})(\bar{f} + g - b - \bar{b})F_{58} - \bar{f}(\bar{f} + g)F_{14}}{(\bar{f} + g - b - \bar{b})F_{58} - (\bar{f} + g)F_{14}},$$

and φ^{-1} is given by $(\bar{f}, \bar{g}) \rightarrow (\bar{f}, g) \rightarrow (f, g)$ given by

$$\begin{aligned} g &= -\frac{(\bar{f} - b - \bar{b})(\bar{f} + \bar{g} - 2\bar{b})F_{58} - \bar{f}(\bar{f} + \bar{g})F_{14}}{(\bar{f} + \bar{g} - 2\bar{b})F_{58} - (\bar{f} + \bar{g})F_{14}}, \\ f &= -\frac{(g - 2b)(\bar{f} + g - b - \bar{b})G_{58} - g(\bar{f} + g)G_{14}}{(\bar{f} + g - b - \bar{b})G_{58} - (\bar{f} + g)G_{14}}. \end{aligned}$$

It is easy to see that the indeterminate points of the first map $\bar{f} = \bar{f}(f, g)$ are either given by the conditions $f + g = 2b$ and $G_{14} = 0$, or by the conditions $f + g = 0$ and $G_{58} = 0$. Thus, we get 8 indeterminate points lying on two curves of bi-degree $(1, 1)$, $C_b : f + g = 2b$ and $C_0 : f + g = 0$:

$$p_i(b + b_i, b - b_i), \quad i = 1, \dots, 4 \quad \text{and} \quad p_i(-b_i, b_i), \quad i = 5, \dots, 8$$

on the (f, g) -plane. It is also easy to see that these are also the indeterminate points of all of the other maps (with b changed to \bar{b} for (\bar{f}, \bar{g}) -coordinates). We then get the following blowup diagram describing the Okamoto space of initial conditions $\mathcal{X}_{\mathbf{b}}$ on Figure 7.

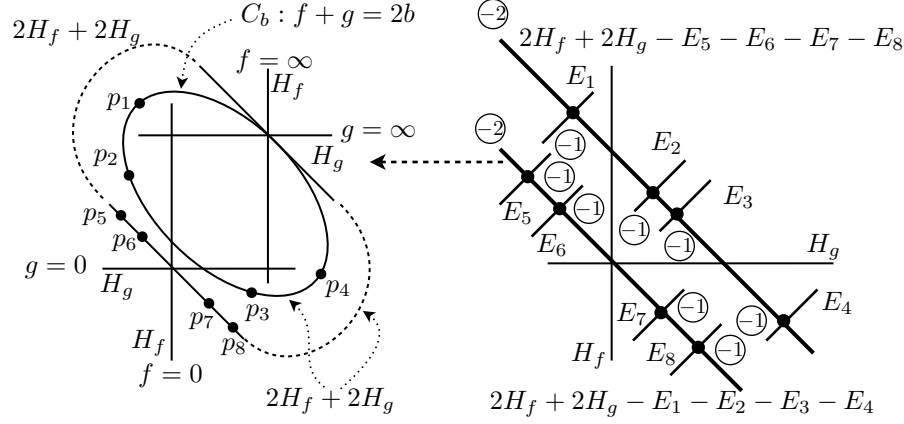


FIGURE 7. Okamoto surface $\mathcal{X}_{\mathbf{b}}$ for the model form of $d-P(A_1^{(1)*})$.

In $\text{Pic}(\mathcal{X}_{\mathbf{b}}) = \mathbb{Z}H_f \oplus \mathbb{Z}H_g \oplus \bigoplus_{i=1}^8 \mathbb{Z}E_i$, the anti-canonical divisor again decomposes uniquely as the sum of two connected components,

$$\begin{aligned} -K_{\mathcal{X}} &= 2H_f + 2H_g - \sum_{i=1}^8 E_i = D_0 + D_1, \quad \text{where} \\ D_0 &= H_f + H_g - E_1 - E_2 - E_3 - E_4, \\ D_1 &= H_f + H_g - E_5 - E_6 - E_7 - E_8, \end{aligned}$$

and $D_1^2 = D_2^2 = -D_1 \bullet D_2 = -2$. Thus, the configuration of components D_i is described by the Dynkin diagram of type $A_1^{(1)}$. To this diagram again correspond two different types of surfaces, the generic one corresponding to divisors D_1 and D_2 intersecting at two points gives a multiplicative system of type $A_1^{(1)}$, and the degenerate configuration corresponding to two components touch at one point gives an additive system denoted by $A_1^{(1)*}$, which is our case, see Figure 8.

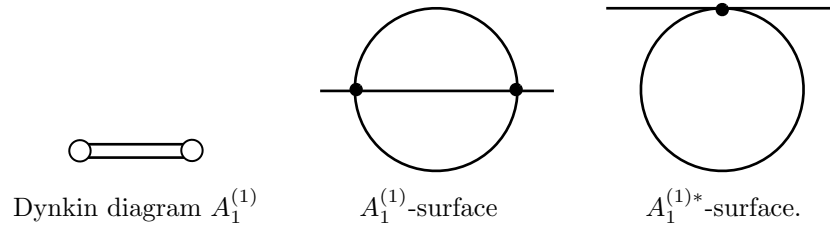


FIGURE 8. Configurations of type $A_1^{(1)}$

The symmetry sub-lattice $R^\perp = \text{Span}_{\mathbb{Z}}\{\alpha_0, \dots, \alpha_7\}$ is of type $E_7^{(1)}$, where the basis of α_i is given on Figure 9.

To compute the action of φ_* on $\text{Pic}(\mathcal{X}_{\mathbf{b}})$, we decompose $\varphi = \varphi_2 \circ \varphi_1$, where $\varphi_1 : (f, g) \rightarrow (\bar{f}, \bar{g})$ is given by (3.10), and $\varphi_2 : (\bar{f}, \bar{g}) \rightarrow (\bar{f}, \bar{g})$ is given by (3.11). Then $(\varphi_1)_*(H_g) = H_g$, it is straightforward to see that $(\varphi_1)_*(E_i) = H_g - E_i$ and that $(\varphi_1)_*(D_0) = D_1$, $(\varphi_1)_*(D_1) = D_0$, which gives $(\varphi_1)_*(H_f) = H_f + 4H_g - E$, where $E = \sum_{i=1}^8 E_i$. The situation with φ_2 is completely symmetric, $(\varphi_2)_*(H_f) = H_f$, $(\varphi_2)_*(E_i) = H_f - E_i$,

$$\begin{array}{ll}
\alpha_1 = E_3 - E_4, & \alpha_5 = E_5 - E_6, \\
\alpha_2 = E_2 - E_3, & \alpha_6 = E_6 - E_7, \\
\alpha_3 = E_1 - E_2, & \alpha_0 = E_7 - E_8, \\
\alpha_4 = H_f - E_1 - E_5, & \alpha_7 = H_g - H_f,
\end{array}$$

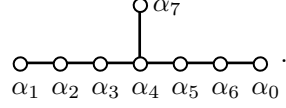


FIGURE 9. Symmetry sub-lattice for $d-P(\tilde{A}_1^*)$

and $(\varphi_2)_*(H_g) = H_g + 4H_f - E$. Composing these two linear maps, we get the action of φ_* :

$$\begin{aligned}
H_f &\mapsto 9H_f + 4H_g - 3E \\
H_g &\mapsto 4H_f + H_g - E \\
E_i &\mapsto 3H_f + H_g - E + E_i, \quad i = 1, \dots, 8,
\end{aligned}$$

and so the induced action φ_* on the sub-lattice R^\perp is given by the following translation:

$$\begin{aligned}
(3.12) \quad (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7) &\mapsto (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7) + \\
&\quad (0, 0, 0, 0, 1, 0, 0, -2)(-K_{\mathcal{X}}).
\end{aligned}$$

3.2.2. Schlesinger Transformations. We start with a 4×4 Fuchsian system of the spectral type 1111, 1111, 22. In this case it is convenient to have all singular points to be finite, since we need to consider two different kinds of elementary Schlesinger transformations — one between two points with non-repeating eigenvalues, and the other when at one point we have an eigenvalue of multiplicity two. As before, we can use scalar gauge transformations to make some of the eigenvalues to vanish, and so, putting $\theta_3 = \theta_3^1 = \theta_3^2$ we take our Riemann scheme to be

$$\left\{ \begin{array}{ccc} z_1 & z_2 & z_3 \\ \theta_1^1 & \theta_2^1 & \theta_3 \\ \theta_1^2 & \theta_2^2 & \theta_3 \\ \theta_1^3 & \theta_2^3 & 0 \\ 0 & \theta_2^4 & 0 \end{array} \right\}, \quad \theta_1^1 + \theta_1^2 + \theta_1^3 + \theta_2^1 + \theta_2^2 + \theta_2^3 + \theta_2^4 + 2\theta_3 = 0.$$

We first consider an elementary Schlesinger transformation $\left\{ \begin{smallmatrix} 1 & 2 \\ 1 & 1 \end{smallmatrix} \right\}$ for which $\bar{\theta}_1^1 = \theta_1^1 - 1$ and $\bar{\theta}_2^1 = \theta_2^1 + 1$. The multiplier matrix for this transformation is

$$\mathbf{R}(z) = \mathbf{I} + \frac{z_1 - z_2}{z - z_1} \mathbf{P}, \quad \text{where } \mathbf{P} = \frac{\mathbf{b}_{2,1} \mathbf{c}_1^{1\dagger}}{\mathbf{c}_1^{1\dagger} \mathbf{b}_{2,1}} \text{ and we put } \mathbf{Q} = \mathbf{I} - \mathbf{P}.$$

Since this transformation does not involve the point z_3 with multiple eigenvalues, the dynamic is again given by equations (2.13–2.19) that now take the form

$$\begin{aligned}
(3.13) \quad \bar{\mathbf{b}}_{1,1} &= \frac{1}{c_1^1} \mathbf{b}_{2,1}, \quad \bar{\mathbf{b}}_{1,j} = \frac{1}{c_1^j} \left(\mathbf{I} - \frac{\mathbf{P}}{\theta_1^1 - \theta_1^j - 1} \left(\mathbf{A}_2 + \frac{z_2 - z_1}{z_3 - z_1} \mathbf{A}_1 \right) \right) \mathbf{b}_{1,j} \quad (j = 2, 3); \\
\bar{\mathbf{b}}_{2,1} &= \frac{1}{c_2^1} \left((\theta_2^1 + 1) \mathbf{I} + \mathbf{Q} \left(\mathbf{I} + \frac{\mathbf{b}_{2,2} \mathbf{c}_2^{2\dagger}}{\theta_2^1 - \theta_2^2 + 1} + \frac{\mathbf{b}_{2,3} \mathbf{c}_2^{3\dagger}}{\theta_2^1 - \theta_2^3 + 1} + \frac{\mathbf{b}_{2,4} \mathbf{c}_2^{4\dagger}}{\theta_2^1 - \theta_2^4 + 1} \right) \times \right. \\
&\quad \left. \left(\mathbf{A}_1 + \frac{z_1 - z_2}{z_3 - z_2} \mathbf{A}_3 \right) \right) \frac{\mathbf{b}_{2,1}}{\mathbf{c}_1^{1\dagger} \mathbf{b}_{2,1}}; \\
\bar{\mathbf{b}}_{2,j} &= \frac{1}{c_2^j} \mathbf{Q} \mathbf{b}_{2,j} \quad (j = 2, 3, 4); \quad \bar{\mathbf{b}}_{3,j} = \frac{1}{c_3^j} \mathbf{R}(z_3) \mathbf{b}_{3,j} \quad (j = 1, 2); \\
\bar{\mathbf{c}}_1^{1\dagger} &= c_1^1 \frac{\mathbf{c}_1^{1\dagger}}{\mathbf{c}_1^{1\dagger} \mathbf{b}_{2,1}} \left((\theta_1^1 - 1) \mathbf{I} + \left(\mathbf{A}_2 + \frac{z_2 - z_1}{z_3 - z_1} \mathbf{A}_3 \right) \times \right. \\
&\quad \left. \left(\mathbf{I} + \frac{\mathbf{b}_{1,2} \mathbf{c}_1^{2\dagger}}{\theta_1^1 - \theta_1^2 - 1} + \frac{\mathbf{b}_{1,3} \mathbf{c}_1^{3\dagger}}{\theta_1^1 - \theta_1^3 - 1} \right) \mathbf{Q} \right), \quad \bar{\mathbf{c}}_1^{j\dagger} = c_1^j \mathbf{c}_1^{j\dagger} \mathbf{Q} \quad (j = 2, 3); \\
\bar{\mathbf{c}}_2^{1\dagger} &= c_2^1 \mathbf{c}_1^{1\dagger}, \quad \bar{\mathbf{c}}_2^{j\dagger} = c_2^j \mathbf{c}_2^{j\dagger} \left(\mathbf{I} - \left(\mathbf{A}_1 + \frac{z_1 - z_2}{z_3 - z_2} \mathbf{A}_3 \right) \frac{\mathbf{P}}{\theta_2^1 - \theta_2^j + 1} \right) \quad (j = 2, 3, 4); \\
\bar{\mathbf{c}}_3^{j\dagger} &= c_3^j \mathbf{c}_3^{j\dagger} \mathbf{R}^{-1}(z_3),
\end{aligned}$$

where c_i^j are again arbitrary non-zero constants.

Similarly to the previous example, we parameterize the matrices as

$$\begin{aligned}
\mathbf{B}_1 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{C}_1^\dagger = \begin{bmatrix} \theta_1^1 & 0 & 0 & \alpha \\ 0 & \theta_1^2 & 0 & \beta \\ 0 & 0 & \theta_1^3 & \gamma \end{bmatrix}, \\
\mathbf{B}_3 &= \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad \mathbf{C}_3^\dagger = \begin{bmatrix} -(x + \theta_3) & 0 & x & \theta_3 \\ 0 & \theta_3 - y & y & 0 \end{bmatrix}, \\
\mathbf{A}_2 &= -(\mathbf{A}_1 + \mathbf{A}_3).
\end{aligned}$$

Using the condition that the eigenvalues of \mathbf{A}_2 are $\theta_2^1, \dots, \theta_2^4$ we get a system of three linear equations on α , β , and γ with coefficients depending on x and y , which gives us rational functions $\alpha(x, y)$, $\beta(x, y)$, and $\gamma(x, y)$ (again, the resulting expressions are quite large and we omit them). Thus, the space of accessory parameters for Fuchsian systems of this type is two-dimensional and x and y are some coordinates on this space.

The resulting mapping $\psi : (x, y) \rightarrow (\bar{x}, \bar{y})$ becomes very complicated (and computing it requires a Computer Algebra System, in our work we have used **Mathematica**) and so we omit equations describing the map. Nevertheless, it is possible to do a complete geometric analysis of the mapping.

We find that the indeterminate points of ψ are

$$\begin{aligned} p_1 & \left(\frac{(\theta_1^1 + \theta_2^1 + \theta_3)(\theta_1^3 + \theta_2^1)}{\theta_1^1 - \theta_1^3}, -\frac{(\theta_1^2 + \theta_2^1 + \theta_3)(\theta_1^3 + \theta_2^1)}{\theta_1^2 - \theta_1^3} \right), \quad p_5(0, 0), \\ p_2 & \left(\frac{(\theta_1^1 + \theta_2^2 + \theta_3)(\theta_1^3 + \theta_2^2)}{\theta_1^1 - \theta_1^3}, -\frac{(\theta_1^2 + \theta_2^2 + \theta_3)(\theta_1^3 + \theta_2^2)}{\theta_1^2 - \theta_1^3} \right), \quad p_6(-\theta_3, \theta_3), \\ p_3 & \left(\frac{(\theta_1^1 + \theta_2^3 + \theta_3)(\theta_1^3 + \theta_2^3)}{\theta_1^1 - \theta_1^3}, -\frac{(\theta_1^2 + \theta_2^3 + \theta_3)(\theta_1^3 + \theta_2^3)}{\theta_1^2 - \theta_1^3} \right), \\ p_4 & \left(\frac{(\theta_1^1 + \theta_2^4 + \theta_3)(\theta_1^3 + \theta_2^4)}{\theta_1^1 - \theta_1^3}, -\frac{(\theta_1^2 + \theta_2^4 + \theta_3)(\theta_1^3 + \theta_2^4)}{\theta_1^2 - \theta_1^3} \right), \end{aligned}$$

as well as the sequence of infinitely close points

$$p_7 \left(\frac{1}{x} = 0, \frac{1}{y} = 0 \right) \leftarrow p_8 \left(\frac{1}{x} = 0, \frac{x}{y} = -\frac{(\theta_1^1 + 1)(\theta_1^2 - \theta_1^3)}{(\theta_1^2 + 1)(\theta_1^1 - \theta_1^3)} \right).$$

Note also that the points p_1, \dots, p_7 all lie on a $(2, 2)$ -curve Q given by the equation

$$(3.14) \quad ((\theta_1^3 - \theta_1^1)x + (\theta_1^3 - \theta_1^1)y)^2 + (\theta_1^1 - \theta_1^2) ((\theta_1^3 - \theta_1^2 - \theta_3)(\theta_1^3 - \theta_1^1)x + (\theta_1^3 - \theta_1^1 - \theta_3)(\theta_1^3 - \theta_1^2)y) = 0.$$

Resolving indeterminate points of this map using blow-ups gives us the Okamoto surface \mathcal{X}_θ pictured on Figure 10. Note that the -2 curves $D_0 = 2H_x + 2H_y - F_1 - F_2 - F_3 - F_4 - F_5 - F_6 - 2F_7$ and $D_1 = F_7 - F_8$ touch at the point with coordinates $\left(\frac{1}{x} = 0, \frac{x}{y} = -\frac{\theta_1^2 - \theta_1^3}{\theta_1^1 - \theta_1^3} \right)$. Thus, we immediately see that this is indeed a surface of type $A_1^{(1)*}$.

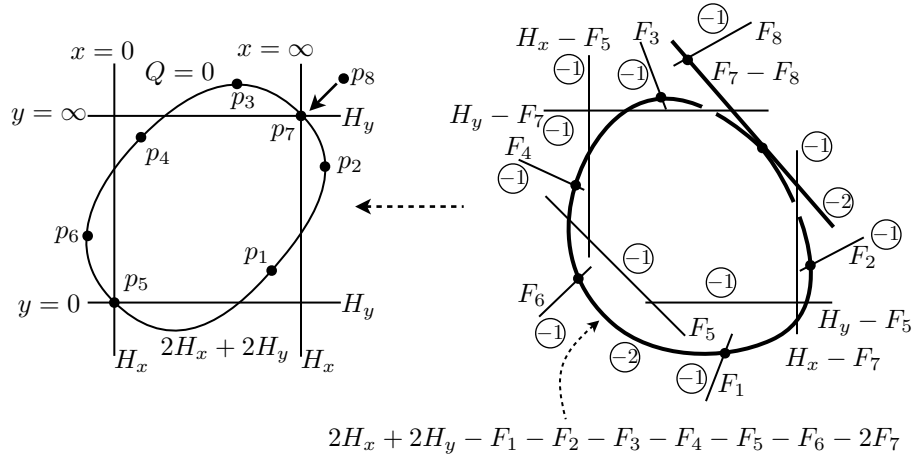


FIGURE 10. Okamoto surface \mathcal{X}_θ for the Schlesinger transformations reduction to $d-P(A_1^{(1)*})$.

3.2.3. Reduction to the standard form. We now proceed to match the surface \mathcal{X}_θ described by the blow-up diagram on Figure 10 with the surface \mathcal{X}_b described by diagram on Figure 7. As in the previous example, we look for rational classes $\mathcal{H}_f, \mathcal{H}_g, \mathcal{E}_1, \dots, \mathcal{E}_8$ in $\text{Pic}(\mathcal{X}_\theta)$ such that

$$\mathcal{H}_f \bullet \mathcal{H}_g = 1, \quad \mathcal{E}_i^2 = -1, \quad \mathcal{H}_f^2 = \mathcal{H}_g^2 = \mathcal{H}_f \bullet \mathcal{E}_i = \mathcal{H}_g \bullet \mathcal{E}_i = \mathcal{E}_i \bullet \mathcal{E}_j = 0, \quad 1 \leq i \neq j \leq 8,$$

and the resulting configuration matches diagram on Figure 7. This time the (virtual) genus formula $g(C) = (C^2 + K_{\mathcal{X}} \bullet C)/2 + 1$ suggests we see that we should look for classes of rational curves of self-intersection zero among H_x, H_y , or $H_x + H_y - F_i - F_j$ and for classes of rational curves of self-intersection -1 among $F_i, H_x - F_i, H_y - F_i$, or $H_x + H_y - F_i - F_j - F_k$.

It is again convenient to start by comparing the -2 -curves on both diagrams,

$$\begin{aligned} D_0 &= 2H_x + 2H_y - F_1 - F_2 - F_3 - F_4 - F_5 - F_6 - 2F_7 \\ &= \mathcal{H}_f + \mathcal{H}_g - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3 - \mathcal{E}_4, \\ D_1 &= F_7 - F_8 = \mathcal{H}_f + \mathcal{H}_g - \mathcal{E}_5 - \mathcal{E}_6 - \mathcal{E}_7 - \mathcal{E}_8. \end{aligned}$$

Given the uniformity of the coordinates of p_i , $i = 1, \dots, 4$, we see that it makes sense to choose $\mathcal{E}_i = F_i$ for $i = 1, \dots, 4$, and also we can put $\mathcal{E}_8 = F_8$. This results in $\mathcal{H}_f + \mathcal{H}_g = 2H_x + 2H_y - F_5 - F_6 - 2F_7$, which suggests taking $\mathcal{H}_f = H_x + H_y - F_5 - F_7$ and $\mathcal{H}_g = H_x + H_y - F_6 - F_7$ (and so $\mathcal{H}_f \bullet \mathcal{H}_g = 1$ and $\mathcal{H}_f^2 = \mathcal{H}_g^2 = 0$). Then $\mathcal{E}_5 + \mathcal{E}_6 + \mathcal{E}_7 = 2H_x + 2H_y - F_5 - F_6 - 3F_7$, so we take $\mathcal{E}_5 = H_y - F_7$, $\mathcal{E}_6 = H_x - F_7$, and $\mathcal{E}_7 = H_x + H_y - F_5 - F_6 - F_7$. It is not very hard to show that such choice satisfies all of our requirements and moreover, it is essentially unique (up to a permutation of the indices of exceptional divisors). To summarize, we get the following identification:

$$\begin{aligned} \mathcal{H}_f &= H_x + H_y - F_5 - F_7, & \mathcal{E}_1 &= F_1, & \mathcal{E}_3 &= F_3, & \mathcal{E}_5 &= H_y - F_7, \\ \mathcal{H}_g &= H_x + H_y - F_6 - F_7, & \mathcal{E}_2 &= F_2, & \mathcal{E}_4 &= F_4, & \mathcal{E}_6 &= H_x - F_7, \\ \mathcal{E}_7 &= H_x + H_y - F_5 - F_6 - F_7, & \mathcal{E}_8 &= F_8. \end{aligned}$$

Let us now define the base coordinates f and g of the linear systems $|\mathcal{H}_f|$ and $|\mathcal{H}_g|$ that will map the exceptional fibers of the divisors \mathcal{E}_i to the points π_i such that π_1, \dots, π_4 are on a line $f + g = \text{const}$ and π_5, \dots, π_8 are on the line $f + g = 0$. Since the pencil $|\mathcal{H}_f|$ consists of $(1, 1)$ curves on $\mathbb{P}^1 \times \mathbb{P}^1$ passing through $p_5(0, 0)$ and $p_7(\infty, \infty)$,

$$\begin{aligned} |\mathcal{H}_f| &= |H_x + H_y - F_5 - F_7| = \{axy + bx + cy + d = 0 \mid a = d = 0\} \\ &= \{bx + cy = 0\}, \end{aligned}$$

and so we can initially define the base coordinate as $f_1 = y/x$. Similarly, the pencil $|\mathcal{H}_g|$ consists of $(1, 1)$ curves on $\mathbb{P}^1 \times \mathbb{P}^1$ passing through $p_6(-\theta_3, \theta_3)$ and $p_7(\infty, \infty)$, so

$$\begin{aligned} |\mathcal{H}_g| &= |H_x + H_y - F_6 - F_7| = \{axy + bx + cy + d = 0 \mid a = 0, (c - b)\theta_3 + d = 0\} \\ &= \{b(x + \theta_3) + c(y - \theta_3) = 0\}, \end{aligned}$$

and $g_1 = (y - \theta_3)/(x + \theta_3)$. Next we will do a series of affine change of variables to arrange that the points p_i , $i = 1, \dots, 4$, are on a line $f + g = \text{const}$. We have (below $i = 1, \dots, 4$)

$$f_1(\pi_i) = -\frac{(\theta_1^1 - \theta_1^3)(\theta_1^2 + \theta_2^i + \theta_3)}{(\theta_1^2 - \theta_1^3)(\theta_1^1 + \theta_2^i + \theta_3)}, \quad g_1(\pi_i) = -\frac{(\theta_1^1 - \theta_1^3)(\theta_1^2 + \theta_2^i)}{(\theta_1^2 - \theta_1^3)(\theta_1^1 + \theta_2^i)},$$

and therefore, it makes sense to put

$$f_2 = \frac{(\theta_1^2 - \theta_1^3)}{(\theta_1^1 - \theta_1^3)} f_1 + 1, \quad g_2 = \frac{(\theta_1^2 - \theta_1^3)}{(\theta_1^1 - \theta_1^3)} g_1 + 1.$$

We then get

$$f_2(\pi_i) = -\frac{(\theta_1^1 - \theta_1^2)}{(\theta_1^1 + \theta_2^i + \theta_3)}, \quad g_2(\pi_i) = -\frac{(\theta_1^1 - \theta_1^2)}{(\theta_1^1 + \theta_2^i)},$$

and so we put

$$f_3 = \frac{(\theta_1^1 - \theta_1^2)}{f_2}, \quad g_3 = \frac{(\theta_1^1 - \theta_1^2)}{g_2}$$

to get

$$f_3(\pi_i) = \theta_1^1 + \theta_2^i + \theta_3, \quad g_3(\pi_i) = \theta_1^1 + \theta_2^i.$$

Thus, our final change of coordinates is

$$\begin{aligned} f &= f_3 - \theta_1^1 = -\frac{\theta_1^2(\theta_1^3 - \theta_1^1)x - \theta_1^1(\theta_1^2 - \theta_1^3)y}{(\theta_1^3 - \theta_1^1)x - (\theta_1^2 - \theta_1^3)y}, \\ g &= \theta_1^1 - g_3 = \frac{\theta_1^2(\theta_1^3 - \theta_1^1)(x + \theta_3) - \theta_1^1(\theta_1^2 - \theta_1^3)(y - \theta_3)}{(\theta_1^3 - \theta_1^1)(x + \theta_3) - (\theta_1^2 - \theta_1^3)(y - \theta_3)}, \end{aligned}$$

and for $i = 1, \dots, 4$, $f(\pi_i) = f(p_i) = \theta_2^i + \theta_3$, $g(\pi_i) = g(p_i) = -\theta_2^i$ and so these points lie on the line $f + g = \theta_3$. We also get the identification of some of the parameters, $\theta_2^i = b_i - b$ for $i = 1, \dots, 4$, and $\theta_3 = 2b$. It remains to verify that this change of variables puts points π_5, \dots, π_8 on the line $f + g = 0$ and identify the remaining parameters. The exceptional divisor \mathcal{E}_5 in the (x, y) -coordinates corresponds to the line $y = \infty$, and so $(f, g)(\pi_5) = (-\theta_1^1, \theta_1^1)$. Similarly, $(f, g)(\pi_6) = (-\theta_2^1, \theta_1^1)$. The exceptional divisor \mathcal{E}_7 corresponds to the line $x + y = 0$, and so $(f, g)(\pi_7) = (-\theta_1^3, \theta_1^3)$. Finally, \mathcal{E}_8 is given by $x = y = \infty$, $\frac{x}{y} = -\frac{(\theta_1^1+1)(\theta_1^2-\theta_1^3)}{(\theta_1^2+1)(\theta_1^1-\theta_1^3)}$, and so $(f, g)(\pi_8) = (1, -1)$. Thus, we see that indeed π_5, \dots, π_8 lie on the line $f + g = 0$ and the remaining identification between the parameters is $\theta_1^1 = b_5$, $\theta_1^2 = b_6$, $\theta_1^3 = b_7$, and $b_8 = -1$.

We are now in the position to compare the dynamic given by an elementary Schlesinger transformation with the dynamic of our model example of $d-P(A_1^{(1)*})$. As in the previous example, there are two different ways to do so. First, we can compute the corresponding translation vector. It is not very difficult to show that the action of ψ_* of an elementary Schlesinger transformation $\begin{Bmatrix} 1 & 1 \\ 1 & 1 \end{Bmatrix}$ on the classes \mathcal{H}_f , \mathcal{H}_g , and \mathcal{E}_i is

$$\begin{aligned} \mathcal{H}_f &\mapsto 4\mathcal{H}_f + 3\mathcal{H}_g - 3\mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3 - \mathcal{E}_4 - 2\mathcal{E}_6 - 2\mathcal{E}_7 - 2\mathcal{E}_8, \\ \mathcal{H}_g &\mapsto 3\mathcal{H}_f + 4\mathcal{H}_g - 3\mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3 - \mathcal{E}_4 - 2\mathcal{E}_6 - 2\mathcal{E}_7 - 2\mathcal{E}_8, \\ \mathcal{E}_1 &\mapsto \mathcal{E}_5, \\ \mathcal{E}_2 &\mapsto 2\mathcal{H}_f + 2\mathcal{H}_g - 2\mathcal{E}_1 - \mathcal{E}_3 - \mathcal{E}_4 - \mathcal{E}_6 - \mathcal{E}_7 - \mathcal{E}_8, \\ \mathcal{E}_3 &\mapsto 2\mathcal{H}_f + 2\mathcal{H}_g - 2\mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_4 - \mathcal{E}_6 - \mathcal{E}_7 - \mathcal{E}_8, \\ \mathcal{E}_4 &\mapsto 2\mathcal{H}_f + 2\mathcal{H}_g - 2\mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3 - \mathcal{E}_6 - \mathcal{E}_7 - \mathcal{E}_8, \\ \mathcal{E}_5 &\mapsto 3\mathcal{H}_f + 3\mathcal{H}_g - 2\mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3 - \mathcal{E}_4 - 2\mathcal{E}_6 - 2\mathcal{E}_7 - 2\mathcal{E}_8, \\ \mathcal{E}_6 &\mapsto \mathcal{H}_f + \mathcal{H}_g - \mathcal{E}_1 - \mathcal{E}_7 - \mathcal{E}_8, \\ \mathcal{E}_7 &\mapsto \mathcal{H}_f + \mathcal{H}_g - \mathcal{E}_1 - \mathcal{E}_6 - \mathcal{E}_8, \\ \mathcal{E}_8 &\mapsto \mathcal{H}_f + \mathcal{H}_g - \mathcal{E}_1 - \mathcal{E}_6 - \mathcal{E}_7. \end{aligned}$$

Comparing the action of ψ_* with the action of the standard dynamic φ_* given by (3.12) on the symmetry sub-lattice, we see that the translation vectors are different:

$$\begin{aligned} \psi_* : (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7) &\mapsto (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7) + \\ &\quad (0, 0, 0, -1, 0, 1, 0, 0)(-K_{\mathcal{X}}), \\ \varphi_* : (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7) &\mapsto (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7) + \\ &\quad (0, 0, 0, 0, 1, 0, 0, -2)(-K_{\mathcal{X}}). \end{aligned}$$

To get more insight into the relationship between Schlesinger transformations and the standard $d-P(A_1^{(1)*})$ dynamic, it is better to compute the action of $d-P(A_1^{(1)*})$ on the Riemann scheme of our Fuchsian system

using the above identification of parameters:

$$\begin{aligned} \begin{pmatrix} z_1 & z_2 & z_3 \\ \theta_1^1 & \theta_2^1 & \theta_3 \\ \theta_1^2 & \theta_2^2 & \theta_3 \\ \theta_1^3 & \theta_2^3 & 0 \\ 0 & \theta_2^4 & 0 \end{pmatrix} &\xrightarrow{\left\{ \begin{smallmatrix} 1 & 2 \\ 1 & 1 \end{smallmatrix} \right\}} \begin{pmatrix} z_1 & z_2 & z_3 \\ \theta_1^1 - 1 & \theta_2^1 + 1 & \theta_3 \\ \theta_1^2 & \theta_2^2 & \theta_3 \\ \theta_1^3 & \theta_2^3 & 0 \\ 0 & \theta_2^4 & 0 \end{pmatrix}, \\ \begin{pmatrix} z_1 & z_2 & z_3 \\ \theta_1^1 & \theta_2^1 & \theta_3 \\ \theta_1^2 & \theta_2^2 & \theta_3 \\ \theta_1^3 & \theta_2^3 & 0 \\ 0 & \theta_2^4 & 0 \end{pmatrix} &\xrightarrow{\text{d-}P(A_1^{(1)*})} \begin{pmatrix} z_1 & z_2 & z_3 \\ \theta_1^1 & \theta_2^1 - 1 & \theta_3 + 2 \\ \theta_1^2 & \theta_2^2 - 1 & \theta_3 + 2 \\ \theta_1^3 & \theta_2^3 - 1 & 0 \\ 0 & \theta_2^4 - 1 & 0 \end{pmatrix}. \end{aligned}$$

Thus, we see that the standard $\text{d-}P(A_1^{(1)*})$ dynamic changes the multiple eigenvalue and so it requires the use of rank-two elementary Schlesinger transformations. In fact, the action on the Riemann scheme suggests that

$$\text{d-}P(A_1^{(1)*}) = \left\{ \begin{smallmatrix} 2 & 3 \\ 3 & 1 \\ 4 & 2 \end{smallmatrix} \right\} \circ \left\{ \begin{smallmatrix} 2 & 3 \\ 1 & 1 \\ 2 & 2 \end{smallmatrix} \right\}.$$

Thus, consider the transformation $\left\{ \begin{smallmatrix} 2 & 3 \\ 1 & 1 \\ 2 & 2 \end{smallmatrix} \right\}$ changing the characteristic indices by $\bar{\theta}_2^1 = \theta_2^1 - 1$, $\bar{\theta}_2^1 = \theta_2^1 - 1$, and $\bar{\theta}_3 = \theta_3 - 1$. This transformation is given by equations (2.31–2.37) that now take the form

$$\begin{aligned} (3.15) \quad \bar{\mathbf{b}}_{1,j} &= \frac{1}{c_1^j} \mathbf{R}(z_1) \mathbf{b}_{1,j} \quad (j = 1, 2, 3); \quad \bar{\mathbf{c}}_1^{j\dagger} = c_1^j \mathbf{c}_1^{j\dagger} \mathbf{R}^{-1}(z_i) \quad (j = 1, 2, 3); \\ \bar{\mathbf{b}}_{2,1} &= \frac{1}{c_2^1} \mathbf{Q}_2 \mathbf{b}_{3,1}, \quad \bar{\mathbf{b}}_{2,2} = \frac{1}{c_2^2} \mathbf{Q}_1 \mathbf{b}_{3,2}, \\ \bar{\mathbf{b}}_{2,j} &= \frac{1}{c_2^j} \left(\mathbf{I} - \left(\frac{\mathcal{P}_1}{\theta_2^1 - \theta_2^j - 1} + \frac{\mathcal{P}_2}{\theta_2^2 - \theta_2^j - 1} \right) \left(\frac{z_3 - z_2}{z_1 - z_2} \mathbf{A}_1 + \mathbf{A}_3 \right) \right) \mathbf{b}_{2,j} \quad (j = 3, 4); \\ \bar{\mathbf{c}}_2^{1\dagger} &= c_2^1 \frac{\mathbf{c}_2^{1\dagger}}{\mathbf{c}_2^{1\dagger} \mathbf{Q}_2 \mathbf{b}_{3,1}} \left((\theta_2^1 - 1) \mathbf{I} + \left(\frac{z_3 - z_2}{z_1 - z_2} \mathbf{A}_1 + \mathbf{A}_3 \right) \times \right. \\ &\quad \left. \left(\mathbf{I} + \frac{\mathbf{b}_{2,3} \mathbf{c}_2^{3\dagger}}{\theta_2^1 - \theta_2^3 - 1} + \frac{\mathbf{b}_{2,4} \mathbf{c}_2^{4\dagger}}{\theta_2^1 - \theta_2^4 - 1} \right) \mathcal{Q} \right), \\ \bar{\mathbf{c}}_2^{2\dagger} &= c_2^2 \frac{\mathbf{c}_2^{2\dagger}}{\mathbf{c}_2^{2\dagger} \mathbf{Q}_1 \mathbf{b}_{3,2}} \left((\theta_2^2 - 1) \mathbf{I} + \left(\frac{z_3 - z_2}{z_1 - z_2} \mathbf{A}_1 + \mathbf{A}_3 \right) \times \right. \\ &\quad \left. \left(\mathbf{I} + \frac{\mathbf{b}_{2,3} \mathbf{c}_2^{3\dagger}}{\theta_2^2 - \theta_2^3 - 1} + \frac{\mathbf{b}_{2,4} \mathbf{c}_2^{4\dagger}}{\theta_2^2 - \theta_2^4 - 1} \right) \mathcal{Q} \right), \\ \bar{\mathbf{c}}_2^{j\dagger} &= c_2^j \mathbf{c}_2^{j\dagger} \mathcal{Q} \quad (j = 2, 3); \\ \bar{\mathbf{b}}_{3,1} &= \frac{1}{c_3^1} \left((\theta_3 + 1) \mathbf{I} + \mathcal{Q} \left(\frac{z_2 - z_3}{z_1 - z_3} \mathbf{A}_1 + \mathbf{A}_2 \right) \right) \frac{\mathbf{b}_{3,1}}{\mathbf{c}_2^{1\dagger} \mathbf{Q}_2 \mathbf{b}_{3,1}}, \\ \bar{\mathbf{b}}_{3,2} &= \frac{1}{c_3^2} \left((\theta_3 + 1) \mathbf{I} + \mathcal{Q} \left(\frac{z_2 - z_3}{z_1 - z_3} \mathbf{A}_1 + \mathbf{A}_2 \right) \right) \frac{\mathbf{b}_{3,2}}{\mathbf{c}_2^{2\dagger} \mathbf{Q}_1 \mathbf{b}_{3,2}}; \\ \bar{\mathbf{c}}_3^{1\dagger} &= c_3^1 \mathbf{c}_2^{1\dagger} \mathbf{Q}_2, \quad \bar{\mathbf{c}}_3^{2\dagger} = c_3^2 \mathbf{c}_2^{2\dagger} \mathbf{Q}_1, \end{aligned}$$

where

$$\begin{aligned}
\mathbf{P}_1 &= \frac{\mathbf{b}_{3,1}\mathbf{c}_2^{1\dagger}}{\mathbf{c}_2^{1\dagger}\mathbf{b}_{3,1}}, \quad \mathbf{Q}_1 = \mathbf{I} - \mathbf{P}_1, \quad \mathbf{P}_2 = \frac{\mathbf{b}_{3,2}\mathbf{c}_2^{2\dagger}}{\mathbf{c}_2^{2\dagger}\mathbf{b}_{3,2}}, \quad \mathbf{Q}_2 = \mathbf{I} - \mathbf{P}_2; \\
\mathcal{P}_1 &= \frac{\mathbf{Q}_2\mathbf{P}_1}{\text{Tr}(\mathbf{Q}_2\mathbf{P}_1)}, \quad \mathcal{P}_2 = \frac{\mathbf{Q}_1\mathbf{P}_2}{\text{Tr}(\mathbf{Q}_1\mathbf{P}_2)}, \quad \mathcal{P} = \mathcal{P}_1 + \mathcal{P}_2, \quad \mathcal{Q} = \mathbf{I} - \mathcal{P}; \\
\mathbf{R}(z) &= \mathbf{I} + \frac{z_2 - z_3}{z - z_2}\mathcal{P}.
\end{aligned}$$

The Okamoto surface for this dynamic is the same as before (it depends only on the Fuchsian system rather than a particular transformation), and its action ψ_*^{12} on $\text{Pic}(\mathcal{X}_\theta)$ is given by

$$\begin{aligned}
\mathcal{H}_f &\mapsto 6\mathcal{H}_f + 3\mathcal{H}_g - \mathcal{E}_1 - \mathcal{E}_2 - 3\mathcal{E}_3 - 3\mathcal{E}_4 - 2\mathcal{E}_5 - 2\mathcal{E}_6 - 2\mathcal{E}_7 - 2\mathcal{E}_8, \\
\mathcal{H}_g &\mapsto 3\mathcal{H}_f + 2\mathcal{H}_g - 2\mathcal{E}_3 - 2\mathcal{E}_4 - \mathcal{E}_5 - \mathcal{E}_6 - \mathcal{E}_7 - 2\mathcal{E}_8, \\
\mathcal{E}_1 &\mapsto 3\mathcal{H}_f + 2\mathcal{H}_g - \mathcal{E}_2 - 2\mathcal{E}_3 - 2\mathcal{E}_4 - \mathcal{E}_5 - \mathcal{E}_6 - \mathcal{E}_7 - \mathcal{E}_8, \\
\mathcal{E}_2 &\mapsto 3\mathcal{H}_f + 2\mathcal{H}_g - \mathcal{E}_1 - 2\mathcal{E}_3 - 2\mathcal{E}_4 - \mathcal{E}_5 - \mathcal{E}_6 - \mathcal{E}_7 - \mathcal{E}_8, \\
\mathcal{E}_3 &\mapsto \mathcal{H}_f - \mathcal{E}_4, \\
\mathcal{E}_4 &\mapsto \mathcal{H}_f - \mathcal{E}_3, \\
\mathcal{E}_5 &\mapsto 2\mathcal{H}_f + \mathcal{H}_g - \mathcal{E}_3 - \mathcal{E}_4 - \mathcal{E}_6 - \mathcal{E}_7 - \mathcal{E}_8, \\
\mathcal{E}_6 &\mapsto 2\mathcal{H}_f + \mathcal{H}_g - \mathcal{E}_3 - \mathcal{E}_4 - \mathcal{E}_5 - \mathcal{E}_7 - \mathcal{E}_8, \\
\mathcal{E}_7 &\mapsto 2\mathcal{H}_f + \mathcal{H}_g - \mathcal{E}_3 - \mathcal{E}_4 - \mathcal{E}_5 - \mathcal{E}_6 - \mathcal{E}_8, \\
\mathcal{E}_8 &\mapsto 2\mathcal{H}_f + \mathcal{H}_g - \mathcal{E}_3 - \mathcal{E}_4 - \mathcal{E}_5 - \mathcal{E}_6 - \mathcal{E}_7,
\end{aligned}$$

and the action on the symmetry sub-lattice is

$$\begin{aligned}
\psi_*^{12} : (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7) &\mapsto (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7) + \\
&\quad (0, 0, 1, 0, 0, 0, 0, -1)(-K_{\mathcal{X}}).
\end{aligned}$$

Similarly, the action ψ_*^{34} of the rank-two transformation $\left\{\begin{smallmatrix} 2 & 3 \\ 3 & 1 \\ 4 & 2 \end{smallmatrix}\right\}$ on $\text{Pic}(\mathcal{X}_\theta)$ is given by

$$\begin{aligned}
\mathcal{H}_f &\mapsto 6\mathcal{H}_f + 3\mathcal{H}_g - 3\mathcal{E}_1 - 3\mathcal{E}_2 - \mathcal{E}_3 - \mathcal{E}_4 - 2\mathcal{E}_5 - 2\mathcal{E}_6 - 2\mathcal{E}_7 - 2\mathcal{E}_8, \\
\mathcal{H}_g &\mapsto 3\mathcal{H}_f + 2\mathcal{H}_g - 2\mathcal{E}_1 - 2\mathcal{E}_2 - \mathcal{E}_5 - \mathcal{E}_6 - \mathcal{E}_7 - 2\mathcal{E}_8, \\
\mathcal{E}_1 &\mapsto \mathcal{H}_f - \mathcal{E}_2, \\
\mathcal{E}_2 &\mapsto \mathcal{H}_f - \mathcal{E}_1, \\
\mathcal{E}_3 &\mapsto 3\mathcal{H}_f + 2\mathcal{H}_g - 2\mathcal{E}_1 - 2\mathcal{E}_2 - \mathcal{E}_4 - \mathcal{E}_5 - \mathcal{E}_6 - \mathcal{E}_7 - \mathcal{E}_8, \\
\mathcal{E}_4 &\mapsto 3\mathcal{H}_f + 2\mathcal{H}_g - 2\mathcal{E}_1 - 2\mathcal{E}_3 - \mathcal{E}_3 - \mathcal{E}_5 - \mathcal{E}_6 - \mathcal{E}_7 - \mathcal{E}_8, \\
\mathcal{E}_5 &\mapsto 2\mathcal{H}_f + \mathcal{H}_g - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_6 - \mathcal{E}_7 - \mathcal{E}_8, \\
\mathcal{E}_6 &\mapsto 2\mathcal{H}_f + \mathcal{H}_g - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_5 - \mathcal{E}_7 - \mathcal{E}_8, \\
\mathcal{E}_7 &\mapsto 2\mathcal{H}_f + \mathcal{H}_g - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_5 - \mathcal{E}_6 - \mathcal{E}_8, \\
\mathcal{E}_8 &\mapsto 2\mathcal{H}_f + \mathcal{H}_g - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_5 - \mathcal{E}_6 - \mathcal{E}_7,
\end{aligned}$$

and its action on the symmetry sub-lattice is

$$\begin{aligned}
\psi_*^{34} : (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7) &\mapsto (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7) + \\
&\quad (0, 0, -1, 0, 1, 0, 0, -1)(-K_{\mathcal{X}}).
\end{aligned}$$

Thus,

$$\begin{aligned}\psi_*^{34} \circ \psi_*^{12} : (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7) &\mapsto (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7) + \\ &\quad (0, 0, 1, 0, 0, 0, 0, -1)(-K_{\mathcal{X}}) + \\ &\quad (0, 0, -1, 0, 1, 0, 0, -1)(-K_{\mathcal{X}}) \\ &= (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7) + \\ &\quad (0, 0, 0, 0, 1, 0, 0, -2)(-K_{\mathcal{X}}) = \varphi_*.\end{aligned}$$

Finally, using a Computer Algebra System we can verify by a direct calculation that

$$\mathrm{d}\text{-}P(A_1^{(1)*}) = \left\{ \begin{smallmatrix} 2 & 3 \\ 3 & 1 \\ 4 & 2 \end{smallmatrix} \right\} \circ \left\{ \begin{smallmatrix} 2 & 3 \\ 1 & 1 \\ 2 & 2 \end{smallmatrix} \right\}$$

holds on the level of equations as well.

4. CONCLUSION

In this work we further develop a theory of discrete Schlesinger evolution equations that correspond to elementary Schlesinger transformations of ranks one and two of Fuchsian systems. We showed how to obtain difference Painlevé equations of types $\mathrm{d}\text{-}P(A_2^{(1)*})$ and $\mathrm{d}\text{-}P(A_1^{(1)*})$ as reductions of elementary Schlesinger transformations. We also tried to make our computations very detailed in order to illustrate general techniques on how to study discrete Painlevé equations geometrically.

One interesting observation is that standard examples of difference Painlevé equations of these types in both cases can be represented as compositions of elementary Schlesinger transformations. Thus, Schlesinger dynamic should in principle be simpler and it would be interesting to find a nice and simple form of equations giving this dynamic.

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SCHOOL OF MATHEMATICAL SCIENCES, THE UNIVERSITY OF NORTHERN COLORADO, CAMPUS BOX 122, 501 20TH STREET, GREELEY, CO 80639, USA

E-mail address: adzham@unco.edu

FACULTY OF MARINE TECHNOLOGY, TOKYO UNIVERSITY OF MARINE SCIENCE AND TECHNOLOGY, 2-1-6 ETCHU-JIMA, KOTOKU, TOKYO, 135-8533, JAPAN

E-mail address: takenawa@kaiyodai.ac.jp